Government debt and education subsidies under labor-income taxes in a dynastic model with leisure, fertility, and human capital externalities

Bei Li (li.bei@uwa.edu.au)

Discipline of Economics, Business School, University of Western Australia

Jie Zhang (zhangjiecqu@cqu.edu.cn; ecszj@nus.edu.sg)

School of Economics and Business Administration, Chongqing University, China
and Department of Economics, National University of Singapore, Singapore

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Abstract

We study government debt and education subsidies under labor-income taxes in a dynastic model with leisure, fertility, and human-capital externalities that raise fertility but reduce leisure, labor, and education spending from efficient levels. Under labor-income taxes, government debt raises leisure and reduces education spending and may reduce fertility and raise labor, whereas education subsidization raises education spending and leisure and may reduce fertility and increase labor. Depending on the strength of agents’ taste for leisure relative to the taste for the number and welfare of children, optimal debt may exceed or fall short of optimal education subsidies under labor-income taxes. Numerically, the average 2.93% government deficit over GDP in the US is close to the optimal ratio but the education subsidy is much higher than optimal.

Keywords: Government debt; Education subsidies; Leisure; Fertility; Human-capital externalities; Income taxes

JEL Classifications: H6; I2; J0; O1

Correspondence: Jie Zhang, School of Economics and Business Administration, Chongqing University, Chongqing, China 400030; Email: zhangjiecqu@cqu.edu.cn; Tel: +86 23 65102571; Department of Economics, National University of Singapore, Singapore 117570. Email: ecszj@nus.edu.sg; Tel: +65 6516 6024; Fax: +65 6775 2646.
1. Introduction

Government debt and education subsidies have received a great deal of attention in the literature. The role of government debt remains controversial: It is neutral for capital accumulation in a Ricardian world over an infinite horizon; it reduces labor and investment when debt is repaid by income taxes; and it reduces fertility when altruistic parents anticipate future tax burdens on children and increase bequests to children accordingly. Education subsidies are typically regarded as means for internalizing human capital externalities. Such externalities also cause higher fertility rates and lower leisure and labor than efficient levels, so that education subsidization alone cannot fully internalize the externalities. Socially-optimal government debt and education subsidies under lump-sum taxes have emerged, from which government deficit should be equal to education subsidies so that the pay-as-you-use principle of Musgrave (1959) holds. As lump-sum taxes are rarely observed in practice, what happens to optimal government debt and education subsidies under widely-used income taxes that reduce the returns to labor and education?

In this paper, we study government debt and education subsidies under income taxes in a dynastic model of physical and human capital accumulation with leisure, fertility, and human-capital externalities. In our analytical findings, government debt under labor-income taxes raises leisure and reduces education spending and may reduce fertility and raise labor, whereas education subsidization raises education spending and leisure and may reduce fertility and increase labor. We derive the equilibrium solution for dynastic welfare and use it to derive optimal government debt
and education subsidies under labor-income taxes to internalize externalities.

Education subsidization arises typically from positive spillovers of human capital investment in the literature (e.g., Tamura, 1991, 1996, 2006). Other reasons for education subsidization include consumption externalities (e.g. Bishnu, 2013), capital market imperfections (e.g. Kodde and Ritzen, 1985), and redistribution concerns (e.g. Glomm and Ravikumar, 1992; Bovenberg and Jacobs, 2005). However, the externalities also create additional wedges on the labor-leisure tradeoff as well as on the quantity-quality tradeoff concerning children that cannot be eliminated by education subsidies alone. The present model incorporates these tradeoffs in the analysis of optimal education subsidies financed by government debt and income taxes.

Government debt is found to be useful to correct dynamic inefficiencies or externalities in a life-cycle model of Diamond (1965) with overlapping generations. In a dynastic model of Barro (1974), government debt under lump-sum taxes is neutral through counteracting private intergenerational transfers. Government debt reduces fertility and increases capital intensity by increasing the bequest cost of a child, as in Becker and Barro (1988), Lapan and Enders (1990), and Wildasin (1990). Government debt can also promote growth, through reducing fertility and increasing labor and human capital investment per child, and improve efficiency through internalizing human capital externalities in a dynastic model in Zhang (2006).

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1 Many empirical studies find evidence of human capital externalities for individuals’ earnings through several channels such as ethnic groups, neighborhoods, work places, or state funding of schools; see, e.g., Borjas (1995), Rauch (1993), Davies (2002), and Moretti (2004a, 2004b), among others.
However, these studies with endogenous fertility abstract from the leisure-labor tradeoff. With a leisure-labor tradeoff, however, government debt under labor-income taxes reduces labor and capital stock and increases leisure in Burbidge (1983). Fanti and Spataro (2006) also cast doubt on the role of government debt in Diamond (1965) with a leisure-labor tradeoff.

Indeed, combination of public intergenerational transfer schemes such as government debt or social security with education subsidies is effective in internalizing human capital externality and allows for decentralization of social optimum. In a dynastic model with fertility and without leisure, Zhang (2003) shows socially-optimal education subsidization and government debt under consumption taxation. Yew and Zhang (2013) show optimal social security and education subsidization without leisure. With leisure and fertility, Li and Zhang (2015) derive socially-optimal government deficit that equals the education subsidy under lump-sum taxation. However, whether young workers should finance public pensions for the old and education subsidies for children is questioned in Docquier et al. (2007) in a lifecycle model if the social discount rate is sufficiently low.

The present paper departs from existing studies (especially Li and Zhang, 2015) by using labor-income taxes to repay government debt. Since taxing labor income reduces the time cost of leisure and childrearing and the return to education, government debt and education subsidies can no longer be socially optimal, so that the first-order condition approach does not apply. Some new results emerge from the
present work. The optimal government deficit and education subsidy under the labor-income tax can achieve the efficient allocation of output, but leisure and fertility are still above and labor is below the efficient levels. The equality relationship between government deficit and education subsidy embedded in the benefit principle of optimal taxation does not hold in general due to the distortion of income taxation. The direction in which the optimal deficit and education subsidy may deviate from the benefit principle depends on agents’ preference towards leisure relative to the welfare and the number of children. When the taste for leisure is relatively weak (strong), the optimal ratio of government deficit to output should go above (below) the optimal level of education subsidy. Finally, using widely-used labor-income taxes also allows us to calibrate the model to developed countries for quantitative implications.

The remainder of the paper proceeds as follows. Section 2 introduces the model. Section 3 determines the competitive equilibrium. Section 4 analyzes optimal government policies. Section 5 presents quantitative implications with an application to the US economy. The last section concludes the paper.

2. The model

The economy is inhabited by overlapping generations of children and adults over an infinite horizon. Children make no choices and simply embody their human capital. All adults work and each adult is endowed with one unit of time which can be allocated to labor, leisure, and childrearing. Rearing a child needs $v \in (0,1)$ fixed
units of time so that fertility $n_t$ has an upper bound $1/v$. The mass of the adult population $L_t$ evolves over time by $L_{t+1} = \bar{n}_t L_t$. We distinguish an individual quantity of a variable $x$ from its average quantity per adult (worker) by denoting the latter as $\bar{x}$ (e.g. $\bar{n}$). In equilibrium, we expect $x = \bar{x}$ by symmetry since agents in the same generation are identical.

The utility of an adult, $V(t)$, depends on his own consumption $c_t$, the number of children $n_t$, leisure $z_t$ and average utility per child $V(t + 1)$:

$$V(t) = \ln c_t + \rho \ln n_t + \phi \ln z_t + \alpha V(t + 1),$$  

where $\alpha \in (0,1)$ is the discounting factor on children’s average utility, $\rho > 0$ is the taste for utility from the number of children, and $\phi > 0$ is the taste for utility from leisure. The logarithmic utility function is used in order to obtain a reduced-form solution for the dynamic path and the welfare function to determine optimal government debt and education subsidization under income taxes here.

When fertility is endogenous, there are several types of assumptions for parental preference that have different implications for efficiency. The parental preference in (1) including average utility of children is based on Mill (1848) and is extended from those in Razin and Ben-Zion (1975) and Zhang (2003, 2006) to incorporate leisure. It differs from the Benthamite preference adopted by Becker and Barro (1988) where parental utility depends on the total utility of children, which requires the cost of rearing a child to exceed the discounted wage income of the child for an interior solution. There are also lifecycle preferences including the quality or quantity of children or both in Becker and Lewis (1973) and Eckstein and Wolpin (1985) in which the Welfare Theorems do not apply. For efficiency with endogenous population
growth, see Golosov et al. (2007), Michel and Wigniolle (2007), and Conde-Ruiz et al. (2010).

The production of final output $Y_t$ uses physical capital $K_t$ and effective labor $L_t\bar{L}_t\bar{h}_t$ according to a Cobb-Douglas technology:

$$Y_t = D K_t^\theta (L_t\bar{L}_t\bar{h}_t)^{1-\theta} , \quad (2)$$

where $\bar{I}_t$ and $\bar{h}_t$ are per worker labor supply and human capital, respectively; $D > 0$ is total factor productivity; and $0 < \theta < 1$ is the share parameter on physical capital. Physical capital and human capital depreciate fully in one period that lasts for 30 years in this model. In per worker terms, $\bar{y}_t = Y_t/L_t = D \bar{k}_t^\theta (\bar{I}_t\bar{h}_t)^{1-\theta}$ where $\bar{k}_t = K_t/L_t$ is physical capital intensity.

The human capital of a child depends on parental educational spending $e_t$, parental human capital $h_t$, and the average human capital of the adult population $\bar{h}_t$ according to a Cobb-Douglas technology

$$h_{t+1} = A e_t^\delta (h_t^{\beta}\bar{h}_t^{1-\beta})^{1-\delta} , \quad (3)$$

where $0 < \beta \leq 1$ measures the role of parental human capital relative to average human capital, and $0 < \delta < 1$ measures the role of education spending. For $0 < \beta < 1$, average human capital $\bar{h}_t$ exerts spillovers to every child’s education as in Tamura (1991) and De la Croix and Doepke (2003).\(^2\)

The price of the final good is normalized to unity. The wage rate of effective labor $w_t$ and the rental price of capital $r_t$ are determined competitively as follows:

\[^2\] Human capital externalities in education differ from human capital externalities in production that increase the productivity of each worker as in Lucas (1988). Since both forms of externalities reduce private returns to human capital from the social rate, they should yield similar results.
\[ w_t = (1 - \theta)Dk_t^\theta (\bar{I}_t \bar{H}_t)^{-\theta}, \quad (4) \]
\[ r_t = \theta Dk_{t+1}^{\theta - 1} (\bar{I}_t \bar{H}_t)^{1-\theta}. \quad (5) \]

Each parent devotes \( vn_t \) units of time to rearing children, \( z_t \) to leisure, and the remaining \( l_t = 1 - vn_t - z_t \) to working. At the beginning of adulthood, everyone receives a bequest \( a_t \) plus interest income from his parent. Wage earnings and bequest income are spent on consumption, education for children, and bequests to children:

\[ c_t = a_t r_t + (1 - vn_t - z_t)w_t h_t (1 - \pi_t) + \tau_t - (1 - s_t)e_t n_t - a_{t+1} n_t, \quad (6) \]

where \( s_t \) is an education-subsidy rate, \( \pi_t \) is a labor-income tax rate, and \( \tau_t \) is a net lump-sum transfer (if positive) or tax (if negative).

The government uses labor-income taxes to finance debt repayment, education subsidies, and transfers, and issues one-period bonds once deficit occurs:

\[ \bar{n}_t \bar{b}_{t+1} + \bar{b}_t r_t + s_t \bar{a}_t \bar{n}_t + \tau_t - \pi_t \bar{I}_t w_t \bar{H}_t, \quad (7) \]

Where \( \bar{b} \) is the amount of outstanding debt per adult. Without uncertainties, government bonds and physical capital are perfect substitutes. The capital market clears when:

\[ K_t = L_t (\bar{a}_t - \bar{b}_t). \quad (8) \]

In per adult terms, \( \bar{k}_t = \bar{a}_t - \bar{b}_t \).

**3. The equilibrium**

Let \( \Omega_t \equiv (r(\bar{k}_t), w(\bar{k}_t), \bar{b}_t, \tau_t, \pi_t, s_t) \) be a vector of prices and government policies.
From (1), (3), and (6), the consumer’s problem is formulated as:

\[
V(a_t, h_t, \Omega_t) = \max_{(a_{t+1}, h_{t+1}, n_t, z_t)} \left\{ \ln[a_t r(\bar{k}_t)] + (1 - v n_t - z_t) w(\bar{k}_t) h_t (1 - \pi_t) + \tau_t 
\right.
\]

\[-(1 - s_t) h_t^{1/\delta} \delta^{-1/\delta} \left( h_t^{\beta - 1/\delta} h_t^{1 - \beta} \right)^{(1-\delta)/\delta} n_t - a_{t+1} n_t \right] +
\]

\[\rho \ln n_t + \phi \ln z_t + a V(a_{t+1}, h_{t+1}, \Omega_{t+1}) \}.
\] (9)

The optimal intratemporal condition with respect to fertility equates the marginal rate of substitution between the number of children and consumption to the cost of having one more child in terms of consumption:

\[
\frac{\rho/n_t}{1/c_t} = v w(\bar{k}_t) h_t (1 - \pi_t) + (1 - s_t) e_t + a_{t+1}.
\] (10)

The labor-income tax has a direct negative effect on the time cost of a child, whereas the education subsidy has a direct negative effect on the education cost of a child. Both effects are in favor of higher fertility.

The optimal intratemporal condition with respect to leisure equates the marginal rate of substitution between leisure and consumption to the opportunity cost of leisure in terms of consumption (the forgone after-tax wage income):

\[
\frac{\phi/z_t}{1/c_t} = w(\bar{k}_t) h_t (1 - \pi_t).
\] (11)

There is a direct negative effect of the labor-income tax on the opportunity cost of leisure, which has been captured in the literature with fixed fertility as a channel through which government debt raises leisure and reduces labor and hence reduces capital stock in the long run. However, with endogenous fertility and time-intensive childrearing here, the rise in leisure does not offset labor at a one-for-one rate. When fertility declines, there is more time available for both leisure and labor.
The parental welfare is increasing in initial assets in the envelope condition
\[ V_1(a_t, h_t, \Omega_t) = r(\bar{k}_t) / c_t > 0. \]
The optimal intertemporal condition with respect to bequests equates the marginal rate of substitution between parental consumption today versus children’s average consumption to the relative return on bequests:
\[
\frac{c_{t+1}}{ac_t} = \frac{r(\bar{k}_{t+1})}{n_t}. \tag{12}
\]
The concerned government policies have no direct effects on this intertemporal condition in this dynastic model. As noted in the aforementioned literature, altruistic parents will leave more bequests to children when anticipating higher future tax burdens on children from higher government deficit. The rise in the bequest cost of children may in turn lead to lower fertility.

The parental welfare is also increasing with parental human capital:
\[
V_2(a_t, h_t, \Omega_t) = [(1 - \nu n_t - z_t)w(\bar{k}_t)h_t(1 - \pi_t) + \beta (1 - \delta)(1 - s_t)e_t n_t (1/\delta)]/(c_t h_t) > 0.
\]

Thus, the intertemporal optimal condition with respect to human capital per child equates the marginal rate of substitution between parental consumption and children’s average consumption to the relative return on education spending:
\[
\frac{c_{t+1}}{ac_t} = \frac{\delta (1 - \nu n_{t+1} - z_{t+1})w(\bar{k}_{t+1})h_{t+1}(1 - \pi_{t+1}) + \beta (1 - \delta)(1 - s_{t+1})e_{t+1} n_{t+1}}{(1 - s_t)e_t n_t}. \tag{13}
\]
Here, the future labor-income tax has a direct negative effect on the return to education. Also, the current education subsidy has a direct negative (proportional) effect on the cost of education but the future education subsidy has a direct negative (less-than-proportional) effect on the return to education. Thus, a higher
education-subsidy rate at all times tends to increase education spending.

The transversality conditions associated with assets and human capital are

\[ \lim_{t \to \infty} a^t n_t a_{t+1}/c_t = 0 \quad \text{and} \quad \lim_{t \to \infty} a^t e_t n_t (1 - s_t)/(c_t \delta) = 0, \]

respectively.

**Definition 1.** A competitive equilibrium with an initial state \((k_0, h_0, L_0, b_0)\) is a sequence of allocation \(\{a_{t+1}, k_{t+1}, h_{t+1}, l_t, n_t, z_t, e_t, c_t, y_t\}_{t=0}^\infty\), prices \(\{r_t, w_t\}_{t=0}^\infty\), and government policies \(\{\tau_t, b_{t+1}, s_t, \pi_t\}_{t=0}^\infty\) such that: (i) taking prices and government policies as given, firms and consumers optimize and their solutions satisfy the budget constraint (6), technologies (2) and (3), first-order conditions (4), (5) and (10)-(13), and the transversality conditions; (ii) the government budget constraint (7) holds; (iii) markets clear so that \(\bar{I}_t = 1 - \nu \bar{n}_t - \bar{z}_t\) and (8) hold; (iv) consistency holds: \(x_t = \bar{x}_t\) \(\forall x_t = a_t, k_t, h_t, l_t, n_t, z_t, e_t, c_t, y_t\).

Since in equilibrium we have \(\bar{x}_t = x_t\) for \(x = a, k, h, l, n, z, e, c, y\), we may drop the notation ‘\(\bar{\cdot}\)’ from average quantities per worker in the rest of the analysis. With Cobb-Douglas technologies and logarithmic (homothetic) preference, it is expected that fertility, the allocation of time, and the proportional allocation of output will be constant over time if so are the ratio of deficit to output and the rates of taxes and subsidies. Thus, let \(\Gamma_c \equiv c_t/y_t\), \(\Gamma_k \equiv k_{t+1} n_t/y_t\), and \(\Gamma_e \equiv e_t n_t/y_t\) be the fractions of output consumed and invested in physical capital and human capital, respectively. Similarly, let \(\Gamma_b \equiv b_{t+1} n_t/y_t\) be the ratio of government deficit to output and \(\Gamma_a \equiv a_{t+1} n_t/y_t\) be the fraction of output left as bequests. Then, the transformed
feasibility is $\Gamma_c = 1 - \Gamma_k - \Gamma_e$.

Forcing the lump-sum transfers to be zero and dividing both sides of the government budget constraint (7) by $y_t$ gives rise to:

$$\Gamma_b = \theta \Gamma_b / \Gamma_k + s \Gamma_e - \pi (1 - \theta).$$

(14)

Using (14) to substitute the ratio of deficit to output and the education subsidy for the labor-income tax in (6) and (10)-(13) yields the following equilibrium solutions (where the superscript $CE$ denotes the competitive equilibrium outcome):

$$\Gamma^C_E = \frac{\delta [\alpha (1-\theta) - (1-\alpha) \Gamma_b]}{\alpha \delta s + (1-s)(1-\alpha \beta (1-\delta))},$$

(15)

$$\Gamma^C_E = \alpha \theta,$$

(16)

$$\Gamma^C_a = \alpha \theta + \Gamma_b,$$

(17)

$$\Gamma^C_k = 1 - \Gamma^C_e - \Gamma^C_E,$$

(18)

$$\eta^C_E = \frac{\alpha [\rho \Gamma^C_E - \Gamma_b - \Gamma^C_E (1-s) \Gamma^C_E]}{\nu [\alpha (1-\theta) - \Gamma_b + \alpha (\rho + \phi) \Gamma^C_E - \alpha \Gamma^C_E - \alpha \Gamma^C_E]},$$

(19)

$$z^C_E = \frac{\alpha \phi \Gamma^C_E}{\alpha (1-\theta) - \Gamma_b + \alpha (\rho + \phi) \Gamma^C_E - \alpha \Gamma^C_E - \alpha \Gamma^C_E},$$

(20)

$$l^C_E = \frac{\alpha (1-\theta) - \alpha \Gamma^C_E (1-\alpha) \Gamma_b}{\alpha (1-\theta) - \Gamma_b + \alpha (\rho + \phi) \Gamma^C_E - \alpha \Gamma^C_E - \alpha \Gamma^C_E}. $$

(21)

As shown in Li and Zhang (2015), although the budget-feasible set is not convex (because of the tradeoffs $e_t n_t$ and $a_{t+1} n_t$), the sufficiency of first-order conditions and transversality conditions for optimal choices holds if the taste for the number of children is sufficiently strong (large enough $\rho$). This condition is essentially similar to the second-order condition in Ehrlich and Lui (1991). In this model, the socially optimal allocation is a special case of the laissez-faire solution in the absence of

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3 If the deficit-output ratio were increased permanently in the initial period, the proceeds plus any residual difference between (7) and (14) would be made lump-sum transfers (taxes) in that period, i.e. $\tau_0 \neq 0$; in the subsequent periods $\tau_t = 0$ for $t > 0$. 
externalities ($\beta = 1$). Henceforth, I use the superscript SP to indicate the social planner’s optimal allocation.

**Definition 2.** Let $\Phi \equiv \alpha \theta [1 - \alpha (1 - \delta)] / ((1 - \alpha)^2 [1 - \alpha \theta (1 - \delta)])$, for $\rho \geq \rho \equiv \alpha \delta (1 - \alpha)^{-1} + \Phi (1 - \alpha) (1 - \delta)$, the interior socially optimal allocation is

$$
\Gamma_e^{SP} = \frac{\alpha \delta (1 - \theta)}{1 - \alpha (1 - \delta)}, \quad \Gamma_k^{SP} = \alpha \theta, \quad \Gamma_c^{SP} = 1 - \Gamma_e^{SP} - \Gamma_k^{SP},
$$

$$
\eta^{SP} = \frac{\rho \Gamma_e^{SP} - \Gamma_k^{SP}}{\nu [1 - \theta + (\rho + \phi) \Gamma_e^{SP} - \Gamma_k^{SP}]},
$$

$$
\ell^{SP} = \frac{1 - \theta}{1 - \theta + (\rho + \phi) \Gamma_e^{SP} - \Gamma_k^{SP}},
$$

$$
z^{SP} = \frac{\phi \Gamma_e^{SP}}{1 - \theta + (\rho + \phi) \Gamma_e^{SP} - \Gamma_k^{SP}}.
$$

The key question that motivates the analysis of the optimal debt and education policy is: how does the competitive solution from (15) to (21) without government intervention ($\Gamma_b = s = 0$) and with human capital externalities ($\beta < 1$) compare to the socially optimal solution? One can easily verify that education spending, leisure, and labor are lower, but fertility and consumption spending are higher than efficient levels. The fraction of output invested in physical capital is at its efficient level. This deviation calls for government intervention.

From (15), there is a negative effect of government deficit on the fraction of income spent on education in equilibrium, because the labor-income tax servicing government deficit in (14) reduces the return on education investment. In (16), government deficit has no effect on the fraction of income invested in physical capital.

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4 With $\Gamma_b = s = 0$, partially differentiating competitive equilibrium solution in (15) to (21) with respect to $\beta$ yields $\partial \ell_e^{CE} / \partial \beta > 0$, $\partial \ell_c^{CE} / \partial \beta < 0$, $\partial \Gamma_b^{CE} / \partial \beta = 0$, $\partial z_e^{CE} / \partial \beta < 0$, $\partial z_k^{CE} / \partial \beta > 0$ and $\partial \ell_c^{CE} / \partial \beta > 0$. 

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in the spirit of the Ricardian equivalence. In (17), government deficit has a positive
effect on bequests to children, which may reduce fertility and thus may increase the
ratio of education spending per child to income. From (18), government deficit has a
positive effect on consumption. In (19), government deficit indeed has opposing
direct effects on fertility (a negative effect via the bequest cost and a positive effect
via labor-income taxes), and indirect effects on fertility through affecting
consumption and education spending. In (20), government deficit has a positive direct
effect on leisure via labor-income taxes, and indirect effects through consumption and
education spending. In (21), government deficit also has opposing direct effects and
indirect effects on labor. The overall effects of government deficit are given below
with the proof in Appendix A.

**Proposition 1.** For $0 \leq s < 1$, a rise in the ratio of government deficit to output $\Gamma_b$
under a labor-income tax will

(i) raise leisure and the fraction of output spent on consumption,

(ii) reduce the fraction of output spent on education,

(iii) have no effect on the fraction of output invested in physical capital,

(iv) reduce (raise) fertility if $\rho$ is smaller (greater) than

$$\alpha[1 - \alpha\beta(1 - \delta) + \phi(1 - \delta)(1 - \alpha\beta)]/\{(1 - \alpha)[1 - \alpha\beta(1 - \delta)]\},$$

(v) raise (reduce) labor if $\phi$ is smaller (greater) than $\alpha/(1 - \alpha) - \rho$.

When lump-sum taxes are used in Li and Zhang (2015), government deficit has a
negative effect on fertility and a positive effect on labor without the conditions in (iv) and (v), and no effect on the fraction of output spent on education. Thus, these conditions arise from labor-income taxation. The results also differ from those in Burbidge (1983) where higher government debt reduces labor and long-run capital stock with fixed fertility.

I now proceed to examine how education subsidization affects fertility, allocations of time and income. From (15) and (18), the education subsidy has a positive effect on the fraction of income invested in children’s education and a negative effect on the fraction of income spent on consumption, because the substitution effect of the subsidy dominates the effect from the labor-income tax. The effects of the education subsidy on fertility and time allocation are ambiguous, because the education subsidy and the labor-income tax tip the tradeoff between the quantity and quality of children and the tradeoff between labor and leisure in different directions. The effects of education subsidies are given below with the proof in Appendix B.

**Proposition 2.** For $0 \leq \Gamma_b < \alpha(1 - \theta)/(1 - \alpha)$, a rise in the education-subsidy rate $s$ under a labor-income tax will

(i) reduce the fraction of output spent on consumption,

(ii) raise leisure and the fraction of output spent on education,

(iii) have no effect on the fraction of output invested in physical capital,
(iv) reduce (raise) fertility if $\rho$ is smaller (greater) than 

$$\alpha[1 - \alpha \beta(1 - \delta) + \phi(1 - \delta)(1 - \alpha \beta)] / ([1 - \alpha][1 - \alpha \beta(1 - \delta)])$$.

(v) raise (reduce) labor if $\phi$ is smaller (greater) than $\alpha / (1 - \alpha) - \rho$.

The effects of education subsidies on fertility and labor are similar to those of government deficit. A rise in government deficit or in the education subsidy financed by the labor income tax can reduce fertility when the taste for the number of children $\rho$ is not too strong relative to the taste for the welfare of children $\alpha$. In particular, if leisure were exogenously given ($\phi = 0$), then the condition for government deficit and education subsidy to reduce fertility would be $\rho < \alpha / (1 - \alpha)$. On the other hand, a rise in either the ratio of government deficit to output or in the rate of education subsidies may raise or reduce labor depending on the strength of the taste for leisure relative to the welfare and the number of children. When the taste for leisure is relatively strong, a rise in government deficit or in education subsidy reduces labor. When the taste for leisure is relatively weak, a rise in government deficit or in the education subsidy raises labor and leisure together. Later on, we will further show the strength of this taste for leisure plays a crucial role in determining how the optimal size of government deficit may deviate from the benefit principle in financing education subsidies.

It is worth noting that the opposite effects of government deficit and the education subsidies on output allocation may cancel out each other, and that their similar effects on fertility and time allocation reinforce each other to dominate the
effects of the labor income tax. Therefore, the net effect on welfare depends mainly on the effects of these government policies on fertility and on time allocation.

Since taxing labor income reduces the time cost of leisure and childrearing and the return to education, government debt and education subsidies might no longer be socially optimal. Hence, to pin down the optimal level of government deficit and education subsidy, it is essential to express the solution for the welfare level by working through the entire dynamic paths of this two-sector model. Substituting the solutions in (15) and (21) in $h_{t+1} = A e^\delta h_t^{1-\delta}$ (with equilibrium condition $h_t = \bar{h}_t$), $y_t = D k_t^{\theta} (l_t h_t)^{1-\theta}$ and $k_{t+1} n = \Gamma_k y_t$ gives rise to

$$k_{t+1} = \Gamma_k D k_t^{\theta} (l_t h_t)^{1-\theta} / n,$$  

(22)

$$h_{t+1} = A (\Gamma e / n)^{\delta} D (1 - vn - z)^{\delta} (k_t / h_t l_t)^{\theta^\delta} h_t.$$  

(23)

Denote the ratio of capital to effective labor as $\mu_t = k_t / h_t l_t$. Equations (22) and (23) determine the evolution of the ratio of physical capital to effective labor:

$$\mu_{t+1} = \left[ \frac{\Gamma_k D^{1-\delta}}{n^{1-\delta} A^{\theta^\delta} (1-vn-z)^{\delta}} \right] \mu_t^{\theta(1-\delta)},$$

$$\mu_\infty = \left[ \frac{\Gamma_k D^{1-\delta}}{n^{1-\delta} A^{\theta^\delta} (1-vn-z)^{\delta}} \right]^{1/[1-\theta(1-\delta)]},$$

where $\mu_t$ is globally convergent toward its long-run level $\mu_\infty$ because $0 < \theta (1 - \delta) < 1$. Substituting $\mu_\infty$ into either (22) or (23) for the long run ratio of physical capital to effective labor gives the (steady-state balanced) growth rate of per capita income:

$$G_\infty = \left\{ A^{1-\theta} (\Gamma e / n)^{\delta(1-\theta)} D^{\delta} (\Gamma k / n)^{\theta^\delta} (1 - vn - z)^{\delta(1-\theta)} \right\}^{1/[1-\theta(1-\delta)]}.$$  

The (gross) long-run growth rate depends positively on physical and human capital
investment per child relative to output and labor but negatively on fertility and leisure.

Letting $\Gamma = \theta(1 - \delta)$ and solving the log-linear versions of these equations yields:

$$\ln \mu_t = (1 - \Gamma^t) \ln \mu_\infty + \Gamma^t \ln \mu_0,$$

$$\ln h_t = t \left[ \ln (AD^\delta) + \delta \ln \left( \frac{\Gamma^t_e}{n} \right) + \delta \ln (1 - vn - z) \right] + \delta \theta \left( t - \frac{1 - \Gamma^t}{1 - \Gamma} \right) \ln \mu_\infty + \delta \theta \left( \frac{1 - \Gamma^t}{1 - \Gamma} \right) \ln \mu_0 + \ln h_0$$

where $\ln \mu_0 = \ln(k_0/h_0) - \ln(1 - vn - z)$ and $\mu_0$ is a function of $n$ and $z$.

Starting from any initial stocks of capital $(h_0, k_0)$ and from the decision rule $(n, z, \Gamma_e, \Gamma_k)$, we can track down the entire dynamic path of capital accumulation $(\mu_t, \mu_\infty, h_t)$ for $t \geq 0$.

Substituting the competitive equilibrium solution of $(n^{CE}, z^{CE}, \Gamma_e^{CE}, \Gamma_k^{CE})$ for $(\ln h_t, \ln \mu_t)$ into utility in (1) gives the equilibrium solution for welfare in terms of the initial state and the proportional allocation rules:

$$V(0) = B_0 + B + \Psi \ln \left( \frac{k_0}{h_0} \right) + \left( \frac{1}{1 - \alpha} \right) \ln h_0,$$

where $B_0$ is a constant (independent of the initial state and of the decision rules) and

$$B = \left( \frac{1}{1 - \alpha} \right) \left[ \rho \ln n^{CE} + \phi \ln z^{CE} + \ln (1 - \Gamma_e^{CE} - \Gamma_k^{CE}) \right.\left. + \ln (1 - vn^{CE} - z^{CE}) \right] + \left[ \frac{a\delta}{(1 - \alpha)^2} \right] \left[ \ln \left( \frac{\Gamma_k^{CE}}{n} \right) + \ln (1 - vn^{CE} - z^{CE}) \right]$$

$$+ \frac{a\delta}{(1 - \alpha)^2} \left[ \ln \left( \frac{\Gamma_e^{CE}}{n} \right) + \ln (1 - vn^{CE} - z^{CE}) \right] - \Psi \ln (1 - vn^{CE} - z^{CE}) + \Phi \left[ \ln \Gamma_k^{CE} - \ln n^{CE} - \delta \ln \left( \frac{\Gamma_k^{CE}}{n} \right) - \delta \ln (1 - vn^{CE} - z^{CE}) \right]$$

with

$$\Phi = \frac{1}{1 - \theta(1 - \delta)} \left[ \frac{a\theta(1 - \delta)(1 - \theta)}{(1 - \alpha)[1 - \alpha \theta(1 - \delta)]} + \frac{a\delta \theta}{(1 - \alpha)^2} \right]$$
\[
\begin{align*}
\Psi &= \frac{\theta [1-a(1-\delta)]}{(1-a)[1-\alpha \theta (1-\delta)]} > 0, \\
\psi &= \frac{\alpha \theta [1-a(1-\delta)]}{[1-\alpha \theta (1-\delta)](1-\alpha)^2} > 0.
\end{align*}
\]

The consumer welfare in (24) is fully characterized by the initial state \((h_0, k_0)\), competitive equilibrium solutions to fertility, and the proportional allocations of time and output. From (15)-(21), notice that \(\Gamma_{CE}^{CE}, n^{CE}\) and \(z^{CE}\) are functions of deficit-output ratio and education subsidy rates, hence the term of \(B\) in (24) can be defined as \(B(s, \Gamma_b)\). Note also government policies influence welfare only through the term \(B(s, \Gamma_b)\).

4. Optimal government policies

To derive optimal government policies, the government can now maximize utility in (24) by choosing \((s, \Gamma_b)\) to maximize \(B(s, \Gamma_b)\) subject to the government budget constraint (14). We present the optimal policies with the labor-income tax below and relegate a sketch of a very lengthy proof to Appendix B.

**Proposition 3.** If \(\rho < \frac{a}{1-\alpha}\), the optimal ratio of deficit to output and education subsidy under labor-income taxation \((s^{**}, \Gamma_{b}^{**})\) satisfying \(\partial B(s, \Gamma_b)/\partial s = 0\) and \(\partial B(s, \Gamma_b)/\partial \Gamma_b = 0\) have positive values and achieve the second-best outcome:

(i) Allocations of output are efficient, i.e. \(\Gamma_{CE}^{CE}(s^{**}, \Gamma_{b}^{**}) = \Gamma_{SP}^{SP}\) and \(\Gamma_{CE}^{CE}(s^{**}, \Gamma_{b}^{**}) = \Gamma_{SP}^{SP}\), whereas fertility, leisure and labor are suboptimal, \(n^{CE}(s^{**}, \Gamma_{b}^{**}) > n^{SP}\), \(z^{CE}(s^{**}, \Gamma_{b}^{**}) > z^{SP}\) and \(l^{CE}(s^{**}, \Gamma_{b}^{**}) < l^{SP}\).
(ii) The benefit principle such as $\Gamma_b^{**} = s^{**}\Gamma_e^{SP}$ holds if and only if $\phi = \frac{\alpha}{1-\alpha} - \rho$; otherwise, $\Gamma_b^{**} \geq s^{**}\Gamma_e^{SP}$ if and only if $\phi \leq \frac{\alpha}{1-\alpha} - \rho$.

In part (i), the optimal government deficit and the optimal education subsidy under a labor-income tax can achieve the efficient allocation of output in Definition 2. However, it cannot achieve the efficient levels of fertility and time allocation in the presence of externalities, because the labor-income tax reduces the opportunity cost of leisure and childrearing. Therefore, the resultant fertility and leisure are higher than the efficient levels, leaving less time for labor. Unlike Li and Zhang (2015), replacing the ideal lump-sum taxation by a realistic labor income tax in servicing government deficit and financing education subsidies can no longer decentralize the social optimum as characterized in Definition 2 although the second-best policy $(s^{**}, \Gamma_b^{**})$ has plenty of room to improve welfare. The reason is that they mitigate the wedges from human capital externalities that lead to higher fertility and lower education spending, leisure and labor than efficient levels. In part (ii), the equality relationship between the optimal levels of government deficit and education subsidy embedded in the benefit principle of optimal taxation as in Li and Zhang (2015) under lump-sum taxation does not hold in general due to the distortion of labor income taxation. Interestingly, the direction in which the second best deficit and education policy may deviate from the benefit principle depends on agents’ preference towards leisure relative to the welfare and the number of children. When the taste for leisure is
relatively weak, the optimal ratio of government deficit to output should go above the
optimal level of education subsidy. Intuitively, when labor responds positively to
government deficit and education subsidy, not only education subsidy should be fully
financed by deficit, but even higher levels of deficit than education subsidy spending
are called for to improve welfare. When the taste for leisure is relatively strong and
labor responds negatively to government deficit and education subsidy, the optimal
ratio of government deficit to output should go below the optimal level of education
subsidy. The education subsidization should be financed by a combination of deficit
and labor-income tax.

5. Numerical results

What are the quantitative implications of the model for the optimal sizes of
government deficits and education subsidies relative to output and for fertility and the
allocations of time and output? To this end, we first calibrate the model to the relevant
economic indicators observed in the US economy and then compare the actual
government deficits and education subsidies over GDP with the simulated values from
cases with laissez faire or with optimal government policies.

The key observations of the US economy are reported in Table 1. To better match
the competitive equilibrium allocation to the real data, we take the average value of
these variables for the past four decades to absorb the impact from short-run shocks
(except for the two education-related indicators which start in 1995). With a single
gender in our model, the calibrated fertility rate is 0.981, half of the actual rate of 1.9614 given in Table 1.

In Table 2, the parameterization is chosen to fit the actual average values from the period 1970-2012. Among them, the labor-income share \( 1 - \theta = 0.67 \) gives the value for capital share, \( \theta = 0.33 \). Then, fitting the investment rate \( \Gamma_k = \alpha \theta \) to the recorded average investment rate of 22.09% gives \( \alpha = 0.6694 \). From the reported expenditures on all levels of educational institutions as percentage of GDP and the proportion of public expenditure on educational institutions, we have \( \Gamma_e = 0.0707 \) and \( s = 0.7071 \). Substituting these values and the recorded ratio of government deficit to output \( \Gamma_b = 0.0293 \) into the government budget in (14) yields the labor-income tax rate \( \pi = 9.62\% \), which is quite close to the estimated average labor-income tax rate of 10.45% in McDaniel (2007) after removing the payroll component for social security contribution from the tax rate on labor income for the same period.

A key parameter is the degree of human capital externalities \( \beta \). Most of existing empirical studies gauge the intergenerational transmission of earnings by measuring the intergenerational elasticity in long-run income, i.e. the coefficient on a father’s log long-run income in the equation of his son’s log long-run income. The central tendency of the resulting estimates is about 0.4 or higher (see Solon, 1999, 2002) and is found relatively stable over two decades for cohorts of sons born 1952-1975 in the US (see Lee and Solon, 2009). This intergenerational earnings elasticity corresponds
to $\delta + \beta (1 - \delta)$ in the log linear version of the human capital of a child in the present model. Taking the elasticity as 0.5 and substituting it into (15) yields $\beta = 0.4818$ and $\delta = 0.0352$.

Next, the parameters to be determined are $\rho$, $\phi$ and $\upsilon$. We assume time for rearing a child as $\upsilon = 0.2$. Then, leisure can be derived by subtracting the labor time (measured by the average labor participation rate) and time spent on rearing children from the one unit of time endowment, $z = 1 - 0.2 \times 0.9807 - 0.6475 = 0.1564$. From $\alpha = 0.6694, \theta = 0.33, s = 0.7071, \Gamma_b = 0.0293, \Gamma_k = 0.2209, \Gamma_e = 0.0706$, we choose the taste parameter for the number of children and the taste parameter for leisure as $\rho = 0.6414$ and $\phi = 0.2065$ in accordance with the solutions in (19) and (20). It is worth mentioning that the choice of $\rho = 0.6414$ satisfies $\rho > \underline{\rho}$ ($\underline{\rho} = 0.3614$) that guarantees an interior solution for fertility, which also satisfies the second-order condition for optimal choices as shown in Li and Zhang (2015). Meanwhile, under this set of parameters for $\rho$ and $\phi$, government debt and education subsidies can generate a negative effect on fertility and a positive effect on labor according to Propositions 1 and 2. Lastly, the values of productivity coefficients $A = D = 1.925$ are chosen to generate the documented annual growth rate of per capita GDP at 1.84% per year according to $g = G^{1/30} - 1$.

Table 3 reports the numerical results for government deficits, education subsidies, labor-income taxes, educational spending, fertility, leisure, labor, output growth rate and consumption-equivalent variations in welfare in four cases: (i) the laissez faire; (ii)
socially optimal government deficits and education subsidies with lump-sum taxes in Li and Zhang (2015); (iii) optimal government deficits and education subsidies with labor-income taxes in Proposition 3; and (iv) the average US government deficits and education subsidies over GDP in the period 1970-2012 used for the calibration. The consumption-equivalent variation from each concerned case $B^i$ to the socially-optimal case $B^*$, denoted by $\Delta$, is determined by

$$B^* = B^i + \sum_{t=0}^{\infty} \alpha^t \ln(1 + \Delta),$$

which leads to $\Delta = \exp[(1 - \alpha)(B^* - B^i)] - 1$.

Comparing cases (i) and (ii), the laissez-faire equilibrium has higher fertility and lower education spending, leisure, and labor than the socially optimal levels because of human-capital externalities. As a result, the welfare level in the socially optimal case is equivalent to a 1.42% increase in consumption in all periods above that in the laissez-faire case.

From cases (i) and (iii), the optimal government deficit and education subsidy with labor-income taxes reduce fertility and increase education spending, leisure and labor from the laissez-faire levels, as in Propositions 1 and 2. From cases (ii) and (iii), the optimal government deficit and education subsidy with labor-income taxes are higher than the socially optimal levels with lump-sum taxes. Also, case (iii) has the same socially optimal education spending as a fraction of output, and higher fertility and leisure and lower labor than their socially optimal levels. The magnitudes of such differences are at most moderate. Thus, the welfare level in case (ii) with lump-sum
taxes is only equivalent to a slight 0.03% increase in consumption from that in case (iii) with labor-income taxes. This finding is encouraging given that labor-income taxes are popularly used and that lump-sum taxes are rarely used in developed countries like the United States. Also, with lump-sum taxation, the education subsidy should be entirely financed by government deficit in Li and Zhang (2015) in the spirit of the pay-as-you-use principle in Musgrave (1959), as \( \Gamma_b = s\Gamma_e = 2.17\% \) in case (ii). However, with labor-income taxation, the optimal government deficit exceeds the optimal education subsidy, as \( \Gamma_b = 3.09\% > s\Gamma_e = 2.30\% \) in case (iii). Here, the excess of the government deficit above the education subsidy helps to mitigate the distortion of the labor-income tax.

From cases (i) and (iv), the average US government deficit and education subsidy increases education spending substantially, reduces fertility, and increases leisure and labor from the laissez-faire levels. Such deficit and subsidy levels are much higher than the socially-optimal levels and lead to much higher education spending relative to output than the socially-optimal level in case (ii). Consequently, the welfare level in case (ii) is nearly equivalent to a 0.9% increase in consumption above that in case (iv). Also, the US government deficit is slightly lower than the optimal level in case (iii), whereas the US education subsidy and labor-income tax rates are much higher than the optimal levels in case (iii). Changing the US government deficit, education subsidy, and income tax to their optimal levels in case (iii) can increase welfare by a 0.84% equivalent increase in consumption.
In Figure 1, we vary the ratio of government deficit to output from 0 to 10% and vary the education subsidy rate from 0 to 90% and obtain the subsequent welfare levels under the baseline parameterizations. It is worth nothing that the welfare surface is concave and smooth, peaking at a unique point which corresponds to the optimal policy in case (iii) with labor-income taxes.

6. Conclusion

In this paper we have analyzed government deficit and education subsidization with labor-income taxation in a dynastic model of physical and human capital accumulation with fertility, leisure, and labor, human capital externalities. The labor-income taxation reduces the return to education and the opportunity cost of time for leisure and childrearing. Such tax distortions change the equilibrium effects of government deficit and education subsidies from the case with lump-sum taxation.

From our analytical finding, education subsidization under labor-income taxes increases education spending as in the literature, reduces fertility unless the taste for the number of children is too strong, and raises labor unless the taste for leisure is too strong. Also, government deficit with labor-income taxes raises leisure as opposed to existing results; it reduces the fraction of output for education spending; it reduces fertility unless the taste for the number of children is too strong; and it raises labor unless the taste for leisure is too strong as opposed to existing results.

Also, we have derived optimal government deficit and education subsidization
with labor-income taxes. Although it achieves the same socially optimal level of education spending relative to output, it still yields higher fertility and leisure than the socially optimal levels. Methodologically, labor-income taxation does not allow government deficit and education subsidies to obtain the efficient allocation due to the tax distortion. Thus, we have used specific functions to track down reduced-form solutions for the allocation of time and income and for the entire dynamic path. This enables us to find the equilibrium solution for the welfare function of initial states and government policies and to derive optimal government policies.

From our numerical results calibrated to the US economy, the allocation from the optimal government deficit and education subsidies with labor-income taxes is surprisingly close to that from the optimal policies with lump-sum taxes. The optimal government deficit under labor-income taxes exceeds the optimal education subsidy, from which the pay-as-you-use principle holds. The excess of government deficit over education subsidies mitigates tax distortions. This differs from the case with lump-sum taxes where optimal government deficit should be equal to the education subsidy (the exact pay-as-you-use principle). Also, the average US ratio of government deficit to GDP during 1970-2012 is very close to the optimal ratio with labor-income taxes, whereas the US education subsidy and the labor-income tax rate are much higher than the optimal levels. Reducing the US education subsidy and labor-income rates to the optimal levels can achieve a welfare gain equivalent to a 0.84% increase in consumption.
Appendix A.

Proof of Propositions 1 and 2. From (15)-(21), the responses of the equilibrium solutions for the key variables to a rise in deficit financed by labor income taxes are given below.

\[
\frac{\partial r^E}{\partial \Gamma_b} = - \frac{\partial r^E}{\partial \Gamma_b} = \frac{\delta(1-\alpha)}{\alpha \delta s + (1-s)[1-\alpha \beta(1-\delta)]} > 0 \text{ for } s \in [0,1]. \quad (A.1)
\]

\[
\frac{\partial n^E}{\partial \beta} = \left(\frac{1}{\nu D^2}\right) \left\{ D \left[ \rho \frac{\partial r^E}{\partial \Gamma_b} - 1 - (1-s) \frac{\partial r^E}{\partial \Gamma_b} \right] - [\rho \Gamma^E_c - \Gamma_b - \Gamma^E_k - (1-s)\Gamma^E_E] \right\}.
\]

where \( D \) is defined as \( 1/\alpha \) times the denominator of \( \nu n^E \) and \( z^{CE} \) in (19) and (20). From these partial derivatives and the equilibrium solutions for the proportional allocations of output, we simplify the partial derivative of fertility with respect to the deficit ratio as

\[
\frac{\partial n^E}{\partial \Gamma_b} = \frac{(1-\alpha \theta)(1-s)[(1-\alpha \beta(1-\delta))\rho(1-\alpha) - \alpha \phi(1-\delta)(1-\alpha \beta)]}{\nu D^2 \alpha(\alpha \delta s + (1-s)[1-\alpha \beta(1-\delta)])} < 0 \text{ if } (A.2)
\]

\[
\rho < \alpha [1 - \alpha \beta(1 - \delta)]/[(1 - \alpha)[1 - \alpha \beta(1 - \delta)]]
\]

for \( s \in [0,1] \).

The partial derivatives of leisure and labor with respect to the deficit ratio are

\[
\frac{\partial z^E}{\partial \beta} = \frac{\phi}{\beta^2} \left\{ D \left[ \Gamma^E_c \left\{ (\rho + \phi) \frac{\partial r^E}{\partial \Gamma_b} - \frac{1}{\alpha} \frac{\partial r^E}{\partial \Gamma_b} \right\} \right] \right\} > 0 \text{ for } s \in [0,1]; \quad (A.3)
\]

\[
\frac{\partial r^E}{\partial \Gamma_b} = D^2 \alpha(\alpha \delta s + (1-s)[1-\alpha \beta(1-\delta)]) > 0 \text{ if } \phi < [\alpha/(1 - \alpha)] - \rho \text{ for } s \in [0,1]. \quad (A.4)
\]

Similarly, the responses of these equilibrium solutions to a rise in the education
subsidy rate are derived as below.

\[
\frac{\partial \Gamma_c^{CE}}{\partial s} = - \frac{\partial \Gamma_c^{CE}}{\partial s} = \frac{\delta \Gamma_c^{CE}}{\partial s} = \frac{\delta [s(1-\theta)-(1-\alpha)\Gamma_b][1-\alpha \delta - \alpha \beta(1-\delta)]}{[a\delta s+(1-s)[1-\alpha \beta(1-\delta)]]^2} > 0 \tag{A.5}
\]

for \( 0 \leq \Gamma_b < \frac{\alpha(1-\theta)}{(1-\alpha)} \).

\[
\frac{\partial \Gamma_c^{CE}}{\partial s} = \left( \frac{1}{vD^2} \right) \left\{ D \left[ \rho \frac{\partial \Gamma_c^{CE}}{\partial s} - (1-s) \frac{\partial \Gamma_c^{CE}}{\partial s} + \Gamma_c^{CE} \right] - [\rho \Gamma_c^{CE} - \Gamma_b - \Gamma_k^{CE} - (1-s)\Gamma_k^{CE}] \right\}
\]

\[
= \frac{(\alpha \theta + \Gamma_b)\delta [1-\alpha \beta(1-\delta)]}{vD^2a[a\delta s+(1-s)[1-\alpha \beta(1-\delta)]]^2}
\]

which is negative if \( 0 \leq \Gamma_b < \alpha(1-\theta)/(1-\alpha) \) and

\[
\rho < \alpha \beta(1-\delta) + \phi(1-\delta)(1-\alpha \beta) / [(1-\alpha)[1-\alpha \beta(1-\delta)]].
\]

Furthermore, we have:

\[
\frac{\partial \Gamma_c^{CE}}{\partial s} = \left( \frac{\phi}{D^2} \right) \left\{ D \left[ \rho \frac{\partial \Gamma_c^{CE}}{\partial s} - \Gamma_c^{CE} \right] \right\}
\]

which is positive when \( \phi < [\alpha/(1-\alpha)] - \rho \) and \( 0 \leq \Gamma_b < \alpha(1-\theta)/(1-\alpha) \).

Q.E.D.

Appendix B.

Proof of Proposition 3. Given (24), to maximize welfare, we derive the first-order necessary conditions \( \partial B(s, \Gamma_b)/\partial s = 0 \) and \( \partial B(s, \Gamma_b)/\partial \Gamma_b = 0 \) as follows (we drop the superscript \( CE \) throughout this proof for simplicity):

\[
\frac{\partial B(s, \Gamma_b)}{\partial s} = \frac{1}{1-\alpha} \left[ \frac{\rho \partial n}{n \partial s} + \frac{\phi \partial z}{z \partial s} - \frac{1}{1-\Gamma_e^{CE}} \frac{\partial \Gamma_e^{CE}}{\partial s} - \frac{1}{1-\Gamma_e^{CE}} \left( \frac{\nu \partial n}{\partial s} + \frac{\partial z}{\partial s} \right) \right] +
\]
For convenience, we rewrite them as

\[
\frac{\partial B(s, \Gamma_b)}{\partial s} = \sum_{i=0}^{4} \Pi_i \frac{1}{n} \frac{\partial n}{\partial s} + \Pi_1 \frac{1}{z} \frac{\partial z}{\partial s} + \Pi_2 \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial s} + \Pi_3 \frac{1}{\Gamma_e} \frac{\partial \Gamma_e}{\partial s} + \Pi_4 \frac{1}{\Gamma_d} \frac{\partial \Gamma_d}{\partial s} = 0, \tag{B.1}
\]

\[
\frac{\partial B(s, \Gamma_b)}{\partial \Gamma} = \sum_{i=0}^{4} \Pi_i \frac{1}{n} \frac{\partial n}{\partial \Gamma} + \Pi_1 \frac{1}{z} \frac{\partial z}{\partial \Gamma} + \Pi_2 \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial \Gamma} + \Pi_3 \frac{1}{\Gamma_e} \frac{\partial \Gamma_e}{\partial \Gamma} + \Pi_4 \frac{1}{\Gamma_d} \frac{\partial \Gamma_d}{\partial \Gamma} = 0, \tag{B.2}
\]

where the coefficients of the variables are

\[
\Pi_0 = \frac{\rho}{1 - \alpha} - \frac{a\delta}{(1 - \alpha)^2} + \Phi(1 + \delta), \quad \Pi_1 = \frac{\Phi}{1 - \alpha},
\]

\[
\Pi_2 = \frac{1}{1 - \alpha} + \frac{a\delta}{(1 - \alpha)^2} - \Psi - \Phi\delta = \frac{1 - \delta}{(1 - \alpha)^2},
\]

\[
\Pi_3 = \frac{1}{1 - \alpha}, \quad \Pi_4 = \frac{a\delta}{(1 - \alpha)^2} - \delta\Phi = \frac{a\delta(1 - \theta)}{(1 - \alpha)^2}.
\]

Given the first-order partial derivatives of \( \Gamma_c, \Gamma_e, n, z \) and \( l \) with respect to deficit-output ratio and with respect to subsidy rate as in (A.1) to (A.8), it is obvious to note

\[
\frac{\partial n}{\partial \Gamma_b} = \frac{\partial z}{\partial \Gamma_b} = \frac{\partial l}{\partial \Gamma_b} = \frac{\partial \Gamma}{\partial \Gamma_b} = \frac{\partial \Gamma_e}{\partial \Gamma_b} = \frac{\partial \Gamma_d}{\partial \Gamma_b} = \Delta_1, \quad \text{where} \quad \Delta_1 \equiv \frac{\delta(\alpha(1 - \theta) - (1 - \alpha))}{(1 - \alpha)(1 - \beta)} > 0 \quad \text{for} \quad s \in [0,1] \quad \text{and} \quad \Gamma_b \in [0, \frac{\pi}{(1 - \alpha)}].
\]

In order to solve (B.1) and (B.2) simultaneously, we multiply both sides of (B.2) by \(-\Delta_1\) and add to (B.1):

\[
\Pi_3 \frac{1}{\Gamma_e} \frac{\partial \Gamma_e}{\partial s} + \Pi_4 \frac{1}{\Gamma_e} \frac{\partial \Gamma_e}{\partial \Gamma_b} - \Delta_1 \left( \Pi_3 \frac{1}{\Gamma_e} \frac{\partial \Gamma_e}{\partial \Gamma_b} + \Pi_4 \frac{1}{\Gamma_e} \frac{\partial \Gamma_e}{\partial \Gamma_b} \right) = 0 \quad \Rightarrow
\]

\[
\left( \Pi_3 \frac{\partial \Gamma_e}{\partial s} + \Delta_1 \frac{\partial \Gamma_e}{\partial \Gamma_b} \right) = 0 \quad \text{(with} \quad \frac{\partial \Gamma_e}{\partial s} = - \frac{\partial \Gamma_e}{\partial \Gamma_b} \text{and} \quad \frac{\partial \Gamma_e}{\partial \Gamma_b} = - \frac{\partial \Gamma_e}{\partial s} \text{)} \quad \Rightarrow
\]

\[
\frac{\Pi_4}{\Gamma_e} \frac{\partial \Gamma_e}{\partial s} + \Delta_1 \frac{\partial \Gamma_e}{\partial \Gamma_b} = 0 \quad \text{(with} \quad \frac{\partial \Gamma_e}{\partial s} + \Delta \frac{\partial \Gamma_e}{\partial \Gamma_b} > 0 \Rightarrow \quad \frac{a\delta(1 - \theta)}{(1 - \alpha)(1 - \beta)} = \frac{\Gamma_e}{1 - \alpha - \Gamma_e}.\)
Thus, it is straightforward to solve \( \Gamma_e = \frac{\alpha \delta (1 - \theta)}{[1 - \alpha (1 - \delta)]} = \Gamma_e^{SP} \) and \( \Gamma_c = 1 - \alpha \theta - \Gamma_e = \Gamma_c^{SP} \) under \( s = s^{**} \) and \( \Gamma_b = \Gamma_b^{**} \) which satisfies (B.1) and (B.2) simultaneously. Hence, the welfare-maximizing (optimal) government deficit and education subsidy can deliver the socially-optimal levels of output allocation.

Further, the result of \( \Gamma_e (s^{**}, \Gamma_b^{**}) = \Gamma_e^{SP} \) leads to a positive relationship between the optimal deficit-output ratio and the optimal education subsidy rate:

\[
\Gamma_b^{**} = \frac{\alpha (1 - \theta) [(1 - \alpha \beta (1 - \delta) - \alpha \delta) s^{**} - \alpha (1 - \delta) (1 - \beta)]}{(1 - \alpha)[1 - \alpha (1 - \delta)]},
\]

Evaluating \( \frac{\partial B(s, \Gamma_b)}{\partial \Gamma_b} \) at the point \( (s^{**}, \Gamma_b^{**}) \) according to (B.2) gives:

\[
\begin{align*}
\frac{\partial B(s, \Gamma_b)}{\partial \Gamma_b} &\bigg|_{s=s^{**}, \Gamma_b=\Gamma_b^{**}} = \Pi_0 \frac{\partial \Lambda}{n \partial \Gamma_b} + \Pi_1 \frac{\partial \Lambda}{z \partial \Gamma_b} + \Pi_2 \frac{1}{\Gamma_b} \frac{\partial \Lambda}{\partial \Gamma_b} + \Pi_3 \frac{1}{\Gamma_c} \frac{\partial \Lambda}{\partial \Gamma_c} + \Pi_4 \frac{1}{\Gamma_e} \frac{\partial \Lambda}{\partial \Gamma_e} \\
&= \frac{(1 - \alpha \theta)(1 - s^{**})}{\alpha D(\alpha \delta s^{**} + (1 - s^{**})(1 - \alpha \beta (1 - \delta)))} \left\{ \frac{\rho \Gamma_c^{SP} - \Gamma_c^{SP} - \Gamma_c^{SP} (1 - \alpha \beta (1 - \delta) - \alpha \delta) s^{**} - \rho \Gamma_c^{SP} (1 - \alpha \beta (1 - \delta) - \alpha \delta) s^{**}}{(1 - \alpha \beta (1 - \delta) - \alpha \delta)} \right\} + \frac{\Phi(1 - \alpha \beta (1 - \delta) - \alpha \delta)}{(1 - \alpha \beta (1 - \delta) - \alpha \delta)} + \frac{[1 - \alpha \beta (1 - \delta) (1 - \alpha \beta)](\alpha - \rho + \phi (1 - \alpha))(1 - \theta)}{(1 - \alpha \beta (1 - \delta) - \alpha \delta) (1 - \alpha \beta (1 - \delta))},
\end{align*}
\]

where we have substituted (A.1) to (A.8) to (B.2) and used the result of \( \frac{\Pi_4}{\Gamma_e} - \frac{\Pi_3}{\Gamma_c} = 0 \) to cancel out the last two terms. Given a positive common term outside the bracket, this first order equation indicates \( \Gamma_b^{**} = s^{**} \Gamma_e^{SP} \) holds if and only if \( \phi = \frac{\alpha}{1 - \alpha} - \rho \).

Moreover, if \( \rho < \frac{\alpha}{1 - \alpha} \), it is easy to show \( \phi < \frac{\alpha}{1 - \alpha} - \rho \Rightarrow \Gamma_b^{**} > s^{**} \Gamma_e^{SP} \) and
\[ \phi > \frac{\alpha}{1-\alpha} - \rho \Rightarrow \Gamma_b^* < s^{**}\Gamma_e^{SP}. \]

Next, we will show this optimal policy mix \((s^{**}, \Gamma_b^{**})\) cannot achieve the socially-optimal levels for fertility, leisure, and labor. First, evaluating the equilibrium solution to leisure in (20) at \((s^{**}, \Gamma_b^{**})\), we have

\[
z(s^{**}, \Gamma_b^{**}) = \frac{\phi\Gamma_c^{SP}}{1-\theta-(\Gamma_b^{**}/\alpha)+(\rho+\phi)\Gamma_c^{SP}-\Gamma_e^{SP}-\Gamma_b^{SP}}.
\]

Compare it with the socially-optimal solution in Definition 2 and we have

\[ z(s^{**}, \Gamma_b^{**}) > z^{SP} \]

as long as \(\Gamma_b^{**} > 0\). Similarly, measuring the solution to fertility in (19) at \((s^{**}, \Gamma_b^{**})\), we obtain

\[
n(s^{**}, \Gamma_b^{**}) = \frac{\rho\Gamma_c^{SP}-\Gamma_b^{SP}-(1-s^{**})\Gamma_e^{SP}}{v[1-\theta-(\Gamma_b^{**}/\alpha)+(\rho+\phi)\Gamma_c^{SP}-\Gamma_e^{SP}-\Gamma_b^{SP}]}.
\]

There are two different terms when comparing fertility with its efficient level: one term \(-\Gamma_b^{**}/\alpha\) sitting in the denominator is negative; the other term \(-\Gamma_b^{**} + s^{**}\Gamma_e^{SP}\) in the numerator can be positive (if \(\phi > \frac{\alpha}{1-\alpha} - \rho\)), negative (if \(\phi < \frac{\alpha}{1-\alpha} - \rho\)) or zero (if \(\phi = \frac{\alpha}{1-\alpha} - \rho\)). In the case of \(-\Gamma_b^{**} + s^{**}\Gamma_e^{SP} > 0\) or \(-\Gamma_b^{**} + s^{**}\Gamma_e^{SP} < 0\), it is obvious \(n(s^{**}, \Gamma_b^{**}) > n^{SP}\). When \(-\Gamma_b^{**} + s^{**}\Gamma_e^{SP} < 0\),

\[
n(s^{**}, \Gamma_b^{**}) - n^{SP} = \frac{(\Gamma_b^{**}/\alpha)[\rho\Gamma_c^{SP}-\Gamma_e^{SP}-\Gamma_b^{SP}]-\Gamma_b^{**}-s^{**}\Gamma_e^{SP}[1-\theta-(\Gamma_b^{**}/\alpha)+(\rho+\phi)\Gamma_c^{SP}-\Gamma_e^{SP}-\Gamma_b^{SP}]}{v[1-\theta-(\Gamma_b^{**}/\alpha)+(\rho+\phi)\Gamma_c^{SP}-\Gamma_e^{SP}-\Gamma_b^{SP}][1-\theta-(\Gamma_b^{**}/\alpha)+(\rho+\phi)\Gamma_c^{SP}-\Gamma_e^{SP}-\Gamma_b^{SP}]}.
\]

where \(\rho + \phi < \frac{\alpha}{1-\alpha}\) can ensure a positive numerator so that \(n(s^{**}, \Gamma_b^{**}) - n^{SP} > 0\).

\[ Q.E.D. \]
References:


OECD, various years. OECD Statistics.


World Bank, various years. World Development Indicators.


Table 1. Average values of key economic indicators in the US in 1970-2012

<table>
<thead>
<tr>
<th>Indicator</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fertility</td>
<td>1.9614</td>
</tr>
<tr>
<td>Fiscal deficit (% of GDP)</td>
<td>2.93</td>
</tr>
<tr>
<td>Labor force participation rate (16 years and over, %)</td>
<td>64.75</td>
</tr>
<tr>
<td>Labor income share</td>
<td>0.67</td>
</tr>
<tr>
<td>All levels education spending (% of GDP) since 1995</td>
<td>7.07</td>
</tr>
<tr>
<td>Proportion of public expenditure on education(%) since 1995</td>
<td>70.71</td>
</tr>
<tr>
<td>Gross capital formation (% of GDP)</td>
<td>22.09</td>
</tr>
<tr>
<td>GDP per capita growth (annual %)</td>
<td>1.84</td>
</tr>
</tbody>
</table>

Date sources: fertility, gross capital formation and GDP per capita growth are from World Development Indicators (World Bank), fiscal deficit is from the Economic Report of the President (2015), labor force participation rate is from the US Bureau of Labor Statistics, the labor income share is from OECD Statistics, education spending and the proportion of public expenditure are from OECD (1998-2015).

Table 2. List of baseline parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>Number of years per period</td>
</tr>
<tr>
<td>( t = 30 )</td>
<td>Time rearing a child</td>
</tr>
<tr>
<td>Utility</td>
<td>Discounting factor</td>
</tr>
<tr>
<td>( \alpha = 0.6694 )</td>
<td>Taste for the number of children</td>
</tr>
<tr>
<td>( \rho = 0.6414 )</td>
<td>Taste for leisure</td>
</tr>
<tr>
<td>( \phi = 0.2065 )</td>
<td></td>
</tr>
<tr>
<td>Production of final output</td>
<td>Capital share</td>
</tr>
<tr>
<td>( \theta = 0.33 )</td>
<td></td>
</tr>
<tr>
<td>( A = 1.925 )</td>
<td>Productivity coefficient</td>
</tr>
<tr>
<td>Production of human capital</td>
<td>Share of physical input</td>
</tr>
<tr>
<td>( \delta = 0.0352 )</td>
<td>Degree of externality ( 1 - \beta )</td>
</tr>
<tr>
<td>( \beta = 0.4818 )</td>
<td></td>
</tr>
<tr>
<td>( D = 1.925 )</td>
<td>Productivity coefficient</td>
</tr>
</tbody>
</table>
Table 3. Numerical results

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\Gamma_b$</th>
<th>$s$</th>
<th>$\pi$</th>
<th>$\Gamma_e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$l$</th>
<th>$g^{***}$</th>
<th>$\Delta^{****}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laissez-faire</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>2.29</td>
<td>1.13</td>
<td>0.14</td>
<td>0.62</td>
<td>1.68</td>
<td>1.42</td>
</tr>
<tr>
<td>First-best*</td>
<td>2.17</td>
<td>48.59</td>
<td>0.00</td>
<td>4.46</td>
<td>1.00</td>
<td>0.14</td>
<td>0.65</td>
<td>1.78</td>
<td>--</td>
</tr>
<tr>
<td>Optimal**</td>
<td>3.09</td>
<td>51.52</td>
<td>5.70</td>
<td>4.46</td>
<td>1.00</td>
<td>0.15</td>
<td>0.64</td>
<td>1.78</td>
<td>0.03</td>
</tr>
<tr>
<td>US (average)</td>
<td>2.93</td>
<td>70.71</td>
<td>9.62</td>
<td>7.07</td>
<td>0.98</td>
<td>0.16</td>
<td>0.64</td>
<td>1.84</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Notes: *The first-best case is from Li and Zhang (2015) with lump-sum taxes. **The optimal case is from the present model with labor-income taxes. ***$g$ is annualized per capita output growth rate. **** $\Delta$ is consumption-equivalent welfare losses from concerned cases to the first-best case.

Figure 1. Welfare levels of education-subsidy rates and ratios of deficits to output