# Inference for moments of ratios with robustness against large trimming bias and unknown convergence rate*

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**Abstract**

We consider statistical inference for moments of the form $E[B/A]$, cf. Khan and Tamer (2010). A naïve sample mean is unstable with small denominator, $A$. This paper develops a method of robust inference, and proposes a data-driven practical choice of trimming observations with small $A$. Our sense of the robustness is twofold. First, bias correction allows for robustness against large trimming bias. Second, adaptive inference allows for robustness against unknown convergence rate. The proposed method allows for closer-to-optimal trimming, and more informative inference results in practice. This practical advantage is demonstrated for inverse propensity score weighting through simulation studies.

*Keywords:* bias correction, ratio, robust inference, trimming, unknown convergence rate.

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# 1 Introduction

Moments of ratios of the form $E[B/A]$ are ubiquitous in empirical research. Summary tables often report statistics of ratios of random variables. In addition, there are a number of specific research methods in which moments of ratios are quantities of direct interest, e.g., inverse probability weighting (Horvitz and Thompson, 1952). When some observations have values of the denominator $A$ close to zero, they behave as outliers in terms of the ratio, $B/A$, and thus can exercise large influences on the naïve sample mean statistics.

To avoid this problem, researchers often trim away those observations with small values of the denominator variable $A$. For example, the well-cited paper by Crump, Hotz, Imbens and Mitnik (2009) proposes to trim away observations with the denominator less than 0.1 for estimating average treatment effects. Trimming indeed mitigates the potentially large variance, but it does so at the cost of increased bias in general. Furthermore, trimmed estimators of moments of ratios are known to be associated with an unknown convergence rate.[1] Ideally, we want a method of inference to be robust against these two issues, namely trimming bias and unknown convergence rate. In this paper, we develop such a method of inference for moments of ratios with the twofold robustness. Our assumptions are simple, easily verifiable with concrete function spaces, and general enough to encompass a wide spectrum of models including those cases where a naïve sample mean converges slowly and a trimming improves the convergence rate – see Section 2.5 ahead.

To achieve the first sense of robustness, i.e., against large trimming bias, we need to carefully choose a trimming threshold and to appropriately remove the trimming bias to the extent where the bias no longer affects the center of the asymptotic distribution relative to the variance. We develop a method of bias-corrected inference by estimating and removing the trimming bias to the necessary extent, and accordingly propose a practical and systematic choice of trimming to meet the theoretical requirements. We are not the first to take this approach - this idea is motivated by Calonico, Cattaneo and Titiunik (2014).

The second sense of robustness we aim to achieve is against an unknown convergence rate. The asymptotic variance of trimmed estimators for moments of ratios converges at the

[1] A large body of the literature discusses asymptotic distribution theories for trimmed sums – see Griffin and Pruitt (1987); Csörgo, Haeusler, and Mason (1988); Griffin and Pruitt (1989) and references therein.

parametric $\sqrt{n}$ rate in "regular" cases, whereas its convergence rate can be as slow as the nonparametric rates in "irregular" cases – see Khan and Tamer (2010). Inference should be robustly valid without a prior knowledge of a researcher about whether the case is regular or irregular. In order to accommodate this issue, we employ the rate-adaptive inference method based on self-normalized processes (cf. Peña, Lai, and Shao, 2009), and extend it with the aforementioned bias correction approach to acquire the twofold robustness.

Romano and Wolf (1999), Peng (2001, 2004) and Chaudhuri and Hill (2016) discuss inference for the mean without finite second moments as we do in this paper. In particular, our approach of rate-adaptive inference in conjunction with trimming bias correction is closely related to that of Peng (2001) and Chaudhuri and Hill (2016). In fact, by using the information of the ratio structure, our method complements and adds practical values to the preceding idea of Peng (2001) and Chaudhuri and Hill (2016) in a few dimensions. First, we introduce a data-driven selection of trimming threshold in a systematic way and consistently with the asymptotic theory. Second, our method circumvents the need to pre-estimate the tail index. Third, our framework allows for use of larger trimming thresholds which enables faster convergence rates.[2] We emphasize that we actively use the information of the ratio structure, and a direct comparison of advantages and disadvantages between our framework and this heavy tail literature is not straightforward. The bias correction approach based on the local polynomial expression of the bias near 'zero' (as opposed to infinity) is made feasible with our approach to trimming based on the denominator. This aspect of our method is crucial for the aforementioned practical contributions.

**Notations:** $E[X]$ and $Var(X)$ denote the expected value and the variance of random variable $X$, respectively. Their sample counterparts are denoted by $E_n[X] = n^{-1} \sum_{i=1}^{n} X_i$ and $Var_n(X) = E_n[X^2] - E_n[X]^2$. The convergence in probability and the convergence in distribution are denoted by $\to_p$ and $\to_d$, respectively. $\mathbb{1}\{\cdot\}$ denotes the indicator function.

**Outline of the paper:** The rest of this paper is organized as follows. In Section 2, we present main results of our method of inference. In Section 3, we extend the main results to the case of estimating $A$, and present the inverse propensity score weighting as a leading example. In Section 4, we conduct simulation studies. Mathematical proofs and practical

[2]We remark that inference based on self-normalized sum is feasible even without a trimming, but a trimming contributes to faster convergence rate which is crucial for finite sample performance in practice.

guidelines are delegated to the appendix.

## 2 Main Results

Given an i.i.d. sample of random vector $(A, B)$, consider the problem of estimating

$$\theta = E\left[\frac{B}{A}\right]. \tag{2.1}$$

This estimand $\theta$ exists under part (i) of the following assumption on finite moments.

**Assumption 1.** *(i)* $E\left[|\frac{B}{A}|\right] < \infty$. *(ii)* $E\left[B^4\right] < \infty$. *(iii)* $Var\left(\frac{B}{A}\right) \neq 0$.

Part (ii) states that the possibility of infinite $Var(B/A)$ is imputed to small $A$ rather than heavy-tailed distribution of $B$. Part (iii) assumes away the trivial case of degenerate data where statistical inference is not feasible. For simplicity of writing, we consider the case where $A$ is supported in the half line $\mathbb{R}_+$, although this restriction is not crucial for the substance of the main results. Because the integrand may take large values if $A$ is close to zero, we consider the regularized estimator defined by

$$\widehat{\theta}_h = E_n\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right] \tag{2.2}$$

with a trimming threshold value $h > 0$. The idea behind this estimator is to trim away those observations that have very small values of $A$ in the denominator of the estimand. In Section 3, we present alternative trimming approaches. For a fixed trimming threshold $h$, the mean of the regularized estimator is

$$\theta_h = E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right], \tag{2.3}$$

which exists under Assumption 1 (i). The difference, $\theta_h - \theta$, is the bias of the regularized estimator $\widehat{\theta}_h$ for the true estimand $\theta$, which we will hereafter refer to as a trimming bias. The order of this trimming bias depends on specific applications, but it may well be as slow as the linear order of $h$ in many plausible applications.

The main difficulty lies in that a naïve estimate, $-E_n\left[\frac{B}{A} \cdot \mathbb{1}\{A \leq h\}\right]$, for the bias may entail infinite variance. We take advantage of the fact that $-E\left[\frac{B}{A} \cdot \mathbb{1}\{A \leq h\}\right]$ can be approximated by estimable objects with bounded variances. Graham and Powell (2012, p. 2125) suggests a similar idea in the context of correlated random coefficient panel data models. To the best of our knowledge, however, this idea has not been formally established for robust inference. Moreover, it is unclear whether this idea is generically applicable beyond the framework of Graham and Powell (2012).

Before proceeding with our formal results, we briefly describe the intuition behind our approach. Assumption 1 (i) and suitable regularity conditions imply $E\left[B \mid A=0\right] \cdot f_A(0) = 0$, because we can write $E\left[\frac{B}{A}\right] = \int_0^\infty \frac{E[B|A=a]\cdot f_A(a)}{a} da$. Thus, the opposite of the bias, $E\left[\frac{B}{A} \cdot \mathbb{1}\{A \le h\}\right]$, can be approximated as

$$E\left[\frac{B}{A} \cdot \mathbb{1}\{A \le h\}\right] = \int_0^h \frac{E\left[B \mid A=a\right] \cdot f_A(a)}{a} da$$
$$= \int_0^h \frac{E\left[B \mid A=a\right] \cdot f_A(a) - E\left[B \mid A=0\right] \cdot f_A(0)}{a} da \approx h \cdot \frac{d}{da} E\left[B \mid A=a\right] \cdot f_A(a)\Big|_{a=0}.$$

Furthermore, the derivative $\frac{d}{da} E\left[B \mid A=a\right] \cdot f_A(a)\Big|_{a=0}$ in the last expression can be estimated by the derivative of the numerator of the Nadaraya-Watson estimator whose statistical properties are well studied – see Ullah and Vinod (1993, p. 94) and references therein. For robust inference, we adjust the asymptotic variance by incorporating the variability of the approximate bias estimator following the idea of Calonico, Cattaneo and Titiunik (2014).

## 2.1 Bias Correction

This subsection characterizes the trimming bias for the purpose of conducting valid inference. The bias characterization is based on the smoothness of the density $f_A$ of $A$ as well as the smoothness of the conditional expectation function $E[B|A = \ \cdot\ ]$ as concretely suggested by Assumption 2 below. We introduce the short-hand notation

$$\tau_j(a) = f_A(a) \cdot E[B^j \mid A = a] \tag{2.4}$$

for $j \in \mathbb{N}$.

**Assumption 2** (Smoothness)**.** *(i) The distribution of $A$ is absolutely continuous in a neighborhood of 0. (ii) $\tau_1$ is $k$-times continuously differentiable with a bounded $k$-th derivative in a neighborhood of 0 for an integer $k > 1$. (iii) $\tau_2$, $\tau_3$, and $\tau_4$ are continuously differentiable with bounded first derivatives in a neighborhood of 0.*

One may conceivably feel that Assumption 2 seems strong and may rule out the case where trimming matters, but this is not the case.[3] We argue in Section 2.5 that this assumption is general and encompasses irregular as well as regular cases as in Khan and Tamer (2010). The following theorem argues that Assumptions 1 (i) and 2 (i)–(ii) allow for bias correction up to the order of $h^k$.

**Lemma 2.1** (Bias Correction)**.** *Suppose that Assumptions 1 (i) and 2 (i)–(ii) are satisfied. For the integer $k > 1$ provided in Assumption 2 (ii),*

$$\theta_h - \theta = P_h^{k-1} + O(h^k)$$

*as $h \to 0$, where $P_h^{k-1}$ is defined by*

$$P_h^{k-1} = -\sum_{\kappa=1}^{k-1} \frac{h^\kappa}{\kappa! \cdot \kappa} \cdot \tau_1^{(\kappa)}(0). \tag{2.5}$$

This lemma shows that the trimming bias, $\theta_h - \theta$, of the estimator (2.2) can be decomposed into an estimable part $P_h^{k-1}$ and a remaining bias of order $h^k$. Hence, if the estimable part $P_h^{k-1}$ can be estimated with a bias of order $h^k$, then substituting such an estimate $\widehat{P}_h^{k-1}$ for $P_h^{k-1}$ can correct the trimming bias of the estimator (2.2) up to the order of $h^k$. In other words, we suggest a bias-corrected estimator

$$\widehat{\theta}_h - \widehat{P}_h^{k-1}$$

with any bias estimator $\widehat{P}_h^{k-1}$ satisfying the following property.

$$E[\widehat{P}_h^{k-1}] = P_h^{k-1} + O(h^k).$$

Section 2.2 presents a concrete suggestion of such an estimator.

## 2.2 Bias Estimation

From (2.5) in the statement of Lemma 2.1, we develop a bias estimator $\widehat{P}_h^{k-1}$ based on an estimator of $\tau_1^{(\kappa)}$, $\kappa \in \{1, ..., k-1\}$. The $\kappa$-th derivative of the function $\tau_1$ defined in (2.4) is estimated by the local derivative estimator

$$\widehat{\tau}_1^{(\kappa)}(0) = E_n\left[\frac{(-1)^\kappa \cdot B}{h^{\kappa+1}} \cdot K^{(\kappa)}\left(\frac{A}{h}\right)\right], \tag{2.6}$$

[3]We explicitly state this remark as we previously received a comment that Assumption 2 seems strong and rules out the case where trimming matters.

where $K$ denotes a kernel function satisfying the following properties.

**Assumption 3** (Kernel)**.** *(i) $K$ has the support $(0,1)$. (ii) $\int_0^1 K(u)du = 1$. (iii) $K$ is $k$-times continuously differentiable with a bounded $k$-th derivative.*

Following (2.5), we consider the correction estimator defined by

$$\widehat{P}_h^{k-1} = -\sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot h^\kappa}{\kappa} \cdot \widehat{\tau}_1^{(\kappa)}(0), \tag{2.7}$$

where the weights $\{\rho_\kappa\}_{\kappa=1}^{k-1}$ are chosen to satisfy

$$\kappa_1 \sum_{\kappa=1}^{k-1} \frac{(-1)^\kappa \cdot \rho_\kappa}{\kappa} \cdot \int_0^1 u^{\kappa_1} K^{(\kappa)}(u)\, du = 1 \text{ for each } \kappa_1 = 1, \ldots, k-1. \tag{2.8}$$

Note that such weights $\{\rho_\kappa\}_{\kappa=1}^{k-1}$ implied by (2.8) are different from the weights $\frac{1}{\kappa!}$ for the population counterpart $P_h^{k-1}$ given in (2.5). It is because the correction estimator $\{\widehat{\tau}_1^\kappa(0)\}_{\kappa=1}^{k-1}$ in (2.6) itself has a bias for the population counterpart $\{\tau_1^\kappa(0)\}_{\kappa=1}^{k-1}$, and we need to correct for these biases of the bias estimator.

The first main theorem of this paper states that the asymptotic order of the bias can be controlled up to $O(h^k)$.

**Theorem 2.1** (Bias Estimation)**.** *If Assumptions 1 (i), 2 (i)–(ii), and 3 are satisfied, then*

$$E\left[\widehat{\theta}_h\right] - E\left[\widehat{P}_h^{k-1}\right] = \theta + O(h^k)$$

*as $h \to 0$.*

## 2.3 Rate-Adaptive Inference

The previous subsection focuses on the asymptotic bias of the bias corrected estimator. The current subsection in turn focuses on the stochastic part. At the end of this subsection, we combine the "bias" part and the "stochastic"part to derive an asymptotic distribution result for robust inference.

Making the dependence of the bandwidth $h$ on the sample size $n$ explicit by $h_n$, we introduce the random variable

$$Z_n = \frac{B}{A} \cdot \mathbb{1}\{A > h_n\} + \sum_{\kappa=1}^{k-1} \frac{(-1)^\kappa \cdot \rho_\kappa \cdot B}{h_n \cdot \kappa} K^{(\kappa)}\left(\frac{A}{h_n}\right). \tag{2.9}$$

Note that (2.2), (2.6), (2.7), and (2.9) yield

$$E_n[Z_n] = \widehat{\theta}_{h_n} - \widehat{P}_{h_n}^{k-1},$$

i.e., the sample mean of $Z_n$ is the bias corrected estimator. We also introduce the notation $\sigma_n^2 = E\left[Z_n^2\right] - E\left[Z_n\right]^2$ for the variance of $Z_n$, and its analog estimate

$$\widehat{\sigma}_n^2 = E_n\left[Z_n^2\right] - E_n\left[Z_n\right]^2.$$

We use the following assumption for the asymptotic normality result.

**Lemma 2.2** (Asymptotic Normality)**.** *If Assumptions 1, 2 (iii), and 3 are satisfied, then*

$$n^{1/2}\widehat{\sigma}_n^{-1}\left(E_n[Z_n] - E[Z_n]\right) \to_d N(0,1)$$

*for $nh_n^2 \to \infty$ as $n \to \infty$.*

Recall that Assumptions 1 (i), 2 (i)–(ii), and 3 are used to obtain the "bias" part in Theorem 2.1. On the other hand, Assumption 1, 2 (iii), and 3 are used to obtain the stochastic or "stochastic" part in Lemma 2.2. Combining Theorem 2.1 and Lemma 2.2 together, we obtain the second main result of this paper below.

**Theorem 2.2** (Asymptotic Normality)**.** *If Assumptions 1, 2, and 3 are satisfied, then*

$$n^{1/2}\widehat{\sigma}_n^{-1}\left(\widehat{\theta}_h - \widehat{P}_h^{k-1} - \theta\right) \to_d N(0,1)$$

*for $nh_n^2 \to \infty$ and $nh_n^{2k} \to 0$ as $n \to \infty$.*

### 2.4 Invariance in Convergence Rate

As in Khan and Tamer (2010), the convergence rate of the regularized estimator (2.2) is unknown in general. The next theorem claims that adding the correction estimator will not slow the convergence rate compared with that of the regularized estimator without bias correction.

**Theorem 2.3** (Invariant Convergence Rate)**.** *If Assumptions 1 (i), (iii), 2, and 3 are satisfied, then*

$$Var(\widehat{\theta}_{h_n} - \widehat{P}_{h_n}^{k-1})/Var(\widehat{\theta}_{h_n}) = O(1)$$

*for $h_n \to 0$ as $n \to \infty$.*

Therefore, the cost of bias correction and robust inference is only the augmented variance, and not a slowed convergence rate.

### 2.5 On Generality of Assumptions 1 and 2

As argued in Section 1, our assumptions are simple, easily verifiable with concrete function spaces, and general enough to encompass a wide spectrum of models including those cases where a naïve sample mean converges slowly and a trimming improves the convergence rate. The current section elaborates on this point. In particular, one may conceivably feel that Assumption 2 seems strong and may rule out the cases of $f_A$ where trimming matters, but we argue that this is not the case.

Consider a simple setting where $B = 1$ and the density function $f_A$ is real analytic, i.e., $f_A$ has the power series representation $f_A(a) = \sum_{\kappa=0}^{\infty} c_\kappa a^\kappa$ for a sequence $\{c_\kappa\}_{\kappa=0}^{\infty}$ of real numbers such that $\sum_{\kappa=1}^{\infty} |c_\kappa| < \infty$. For the parameter $E[B/A]$ of interest to be well-defined, $c_0 = 0$ is required in this representation. Assumption 1 always holds as $c_0 = 0$ – see (D.1) and (D.2) in Appendix D for details. Furthermore, since the radius of convergence of the power series is at least one, $f_A$ is infinitely many times differentiable with bounded derivatives in a neighborhood of zero, which implies that Assumption 2 is satisfied.

This setup, satisfying both Assumptions 1 and 2, is general in the sense that it encompasses both the case of bounded second moment of $B/A$ and the case of unbounded second moment of $B/A$. In fact, $E[B^2/A^2] = \infty$ if and only if $c_1 \neq 0$ – see (D.3) and (D.4) in Appendix D for details. An example of such an analytic function $f_A(a)$ with $c_1 \neq 0$ is the density function for the chi-square distribution with four degrees of freedom.

If $c_1 \neq 0$, then the variance of the trimmed mean $\widehat{\theta}_h = E_n\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]$ is $O(\log(1/h))$ as $h = o(1)$ – see (D.5) in Appendix D for details. This shows that a trimming affects the convergence rate of the trimmed estimator. In summary, this section demonstrates that our assumptions are general enough to cover the cases of $f_A$ where a trimming matters.

# 3 Extension: Generated Random Variables

### 3.1 General Framework

In this section, we consider an extended framework where the denominator of a fraction is generated by a transformation

$$A = g(X, \gamma_0)$$

where $g$ is a known function, $X$ is an observed random vector, and $\gamma_0$ is an unknown preliminary parameter as an element of a set $\Gamma$. The parameter of our interest remains

$$\theta = E\left[\frac{B}{A}\right] = E\left[\frac{B}{g(X,\gamma_0)}\right].$$

To fix the idea, consider, for example, the inverse propensity score weighting (Rosenbaum, 1987), where the propensity score $A$ in the denominator is typically a logit or probit transformation $g(\ \cdot\ ,\gamma_0)$ of observed covariates $X$ – see Section 3.2 for details. We use the short-hand notation $A(\gamma) := g(X,\gamma)$ when the role of $X$ is not crucial in exposition.

In practice, a researcher has to estimate the unknown parameter $\gamma_0$ by $\widehat{\gamma}$. The regularized estimator in this setting is given by $\widehat{\theta}_{h_n}(\widehat{\gamma})$, where the regularized sample mean

$$\widehat{\theta}_{h_n}(\gamma) = E_n\left[\frac{B}{A(\gamma)} \cdot S\left(\frac{A(\gamma)}{h_n}\right)\right] \tag{3.1}$$

is based on a trimming function $S$ satisfying Assumption 3′ below. As in the baseline case, this regularized estimator entails a bias, and we propose to correct this bias by the bias estimator $\widehat{P}^{k-1}_{h_n}(\widehat{\gamma})$, where

$$\widehat{P}^{k-1}_{h_n}(\gamma) = -\sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot h_n^\kappa}{\kappa} \cdot \widehat{\tau}_1^{(\kappa)}(0;\gamma). \tag{3.2}$$

The local derivative estimator used in this bias estimator (3.2) is given by

$$\widehat{\tau}_1^{(\kappa)}(0;\gamma) = E_n\left[\frac{(-1)^\kappa \cdot B}{h_n^{\kappa+1}} \cdot K^{(\kappa)}\left(\frac{A(\gamma)}{h_n}\right)\right] \tag{3.3}$$

similarly to (2.6). The weights $\{\rho_\kappa\}_{\kappa=1}^{k-1}$ used in the bias estimator (3.2) are chosen to satisfy

$$\sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa} \cdot \int_0^1 u^{\kappa_1} K^{(\kappa)}(u)\, du = \frac{1}{\kappa_1} - s^{\kappa_1} \text{ for each } \kappa_1 = 1, \ldots, k-1, \tag{3.4}$$

where $s^\kappa = \int_0^1 u^{\kappa-1} S(u) du$ for each $\kappa = 1, ..., k-1$. Assumption 3′ (iv) below guarantees that $0 \leq s^\kappa \leq \kappa^{-1}$.

The bias-corrected estimator $\widehat{\theta}_{h_n}(\widehat{\gamma}) - \widehat{P}^{k-1}_{h_n}(\widehat{\gamma})$ is succinctly written as $\widehat{\mu}_{h_n}(\widehat{\gamma})$, where

$$\widehat{\mu}_{h_n}(\gamma) = E_n\left[B \cdot \frac{1}{h_n} \cdot \omega\left(\frac{A(\gamma)}{h_n}\right)\right] \qquad \text{and}$$

$$\omega(u) = \frac{1}{u} S(u) + \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa} K^{(\kappa)}(u).$$

We write $\omega^{(1)} = \omega'$, $\widehat{\mu}^{(1)}_{h_n} = \nabla \widehat{\mu}_{h_n}$, $\mu_{h_n}(\gamma) = E\left[\widehat{\mu}_{h_n}(\gamma)\right]$, and $\mu^{(1)}_{h_n} = \nabla \mu_{h_n}$. As we formally show, the influence function for the bias-corrected estimator $\widehat{\mu}_{h_n}(\widehat{\gamma})$ takes the form

$$Z_n = \left( B \cdot \frac{1}{h_n} \cdot \omega\left(\frac{A(\gamma_0)}{h_n}\right) - \mu_{h_n}(\gamma_0) \right) + E\left[ B \cdot \frac{1}{h_n^2} \cdot \omega^{(1)}\left(\frac{A(\gamma_0)}{h_n}\right) \cdot A^{(1)}(\gamma_0)^T \right] \cdot \varphi_0(X),$$

where $A^{(1)}(\gamma) = \nabla_\gamma g(X,\gamma)$, and $\varphi_0$ denotes the influence function for the first-step estimation of $\gamma_0$ – precisely defined in Assumption 1′ (iv) below. Since we do not know some components of the influence function $Z_n$, we estimate it by

$$\widehat{Z}_n = \left( B \cdot \frac{1}{h_n} \cdot \omega\left(\frac{A(\widehat{\gamma})}{h_n}\right) - \widehat{\mu}_{h_n}(\widehat{\gamma}) \right) + E_n\left[ B \cdot \frac{1}{h_n^2} \cdot \omega^{(1)}\left(\frac{A(\widehat{\gamma})}{h_n}\right) \cdot A^{(1)}(\widehat{\gamma})^T \right] \cdot \widehat{\varphi}(X),$$

where $\widehat{\varphi}$ estimates $\varphi_0$. We similarly estimate $E\left[Z_n\right]$ by $\widehat{\sigma}^2 = E_n\left[\widehat{Z}_n^2\right]$.

In order to account for the effect of the first-step estimation of $\gamma_0$, we modify Assumptions 1, 2, and 3 from the baseline setting by the following assumptions – Assumptions 1′ (i), 2′ (i)–(ii), and 3′ (i)–(ii) remain exactly the same as Assumptions 1 (i), 2 (i)–(ii), and 3 (i)–(ii), respectively. All the other parts are extensions or new assumptions.

**Assumption 1′.** (i) $E\left[\left|\frac{B}{A}\right|\right] < \infty$. (ii) $n^{-1/4} \cdot E[Z_n^4]^{1/4} / E[Z_n^2]^{1/2} = o(1)$ as $n \to \infty$. (iii) $E[Z_n^2]$ is bounded away from zero. (iv) $\widehat{\gamma} - \gamma_0 = E_n\left[\varphi_0(X)\right] + o_p\left(n^{-1/2}\right)$ as $n \to \infty$, $E\left[\varphi_0(X)\right] = 0$, and $E\left[\|\varphi_0(X)\|^2\right] < \infty$. (v) $E_n\left[\|\widehat{\varphi}(X) - \varphi_0(X)\|^2\right]^{1/2} = o_p(h_n^2)$.

**Assumption 2′** (Smoothness)**.** (i) The distribution of $A$ is absolutely continuous in a neighborhood of 0. (ii) $\tau_1$ is $k$-times continuously differentiable with a bounded $k$-th derivative in a neighborhood of 0 for an integer $k > 1$. (iii) $A(\,\cdot\,)$ is twice continuously differentiable a.s., $E\left[|B| \cdot \sup_{\gamma\in\Gamma} \left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|\right] < \infty$, $E\left[B^2 \cdot \sup_{\gamma\in\Gamma} \left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|\right] < \infty$, $E\left[|B| \cdot \sup_{\gamma\in\Gamma} \left\|A^{(1)}(\gamma)\right\|\right] < \infty$, and $E\left[|B| \cdot \sup_{\gamma\in\Gamma} \left\|A^{(2)}(\gamma)\right\|\right] < \infty$.

**Assumption 3′** (Kernel and Trimming Functions)**.** (i) $K$ has the support $(0,1)$. (ii) $\int_0^1 K(u)du = 1$. (iii) $K$ is $(k+2)$-times continuously differentiable with a uniformly bounded $(k+2)$-nd derivative. (iv) $S(u) = 0$ for all $u \in (-\infty, 0)$, $S(u) \in [0,1]$ for all $u \in [0,1]$, and $S(u) = 1$ for all $u \in (1,\infty)$. (v) $S$ is twice continuously differentiable and $u \mapsto u^{-j} \cdot S^{(3-j)}(u)$ is uniformly bounded for each $j \in \{1,2,3\}$.

The new and extended parts of these assumptions relative to those in the baseline framework are as follows. Assumption 1′ (ii) replaces the bounded fourth moment condition of Assumption 1 (ii). Assumption 1′ (iii) replaces the non-zero variance condition of

Assumption 1 (iii). Assumption 1′ (iv)–(v) are new conditions we require for the first-step estimator $\widehat{\gamma}$ and the influence function estimators, $\widehat{\varphi}$. We keep these high-level statements for generic applicability, but they are verified with a specific example below. Assumption 2′ (iii) requires smoothness and uniformly bounded moments for $A(\,\cdot\,)$. Assumption 3′ (iii) extends Assumption 3 (iii) by increasing the order of smoothness of two. Parts (iv) and (v) of Assumption 3′ list properties that we require for the trimming function $S$ used in the regularized sample mean (3.1). The following theorem, as an extended counterpart of Theorem 2.2, shows the asymptotic normality of the self-normalized sum.

**Theorem 3.1** (Asymptotic Normality)**.** *If Assumptions 1′, 2′, and 3′ are satisfied, then*

$$n^{1/2}\widehat{\sigma}_n^{-1}\left(\widehat{\mu}_{h_n}(\widehat{\gamma}) - \theta\right) \to_d N(0,1)$$

*for* $nh_n^6 \to \infty$ *and* $nh_n^{2k} \to 0$ *as* $n \to \infty$*.*

### 3.2 Example: Inverse Propensity Score Weighting

In this section, we discuss the inverse propensity score weighting (Rosenbaum, 1987) as an example of the general framework introduced in Section 3.1. Let $D$ denote the binary treatment indicator variable. The outcome $Y$ is produced by $Y = (1-D)\cdot Y_0 + D\cdot Y_1$, where $Y_1$ and $Y_0$ denote the counterfactual outcomes with treatment and without treatment, respectively. Let $W$ denote a vector of observed control variables. A researcher observes a random sample drawn from the distribution of $(Y, D, W)$, but does not observe the counterfactual outcomes, $Y_1$ or $Y_0$. The individual treatment effect is defined by $Y_1 - Y_0$. The parameter of interest is the average treatment effect (ATE) defined by $\theta_0 = E[Y_1 - Y_0]$. Rosenbaum and Rubin (1983) show that this ATE is identified by

$$\theta_0 = E\left[\frac{(2D-1)\cdot Y}{D + (2D-1)\cdot(P(D=1 \mid W) - 1)}\right]$$

under the following assumption.

**Assumption 4.** *(i) There is a vector of covariates, denoted by $W$, such that $D$ is independent of $Y_0$ and $Y_1$ given $W$. (ii) $P(P(D=1 \mid W) = 0$ or $1) = 0$.*

In practice, researchers often estimate the propensity score $P(D = 1 \mid W)$ by parametric models of the form $P(D = 1 \mid W) = \pi(W^T\gamma_0)$ with $\pi(v) = \exp(v)/(1 + \exp(v))$ and unknown parameters $\gamma_0$. We can thus write $\theta_0$ as $E[B/A(\gamma_0)]$, where $A(\gamma) = D + (2D - 1) \cdot (\pi(W^T\gamma) - 1)$ and $B = (2D-1) \cdot Y$. When $\gamma$ is estimated via the maximum likelihood estimator $\widehat{\gamma}$, the influence function is $\varphi_0(X) = E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(D - \pi(W^T\gamma_0))W$ and its estimator is $\varphi_0(X) = E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1}(D-\pi(W^T\widehat{\gamma}))W$.

Since $\theta_0$ can be represented by the generic ratio form $E[B/A(\gamma)]$ of our estimand, our method of inference is applicable to this setting. Appendix H.3 presents a concrete implementation procedure for the popular case of the logit propensity score model $\pi(W^T\gamma_0)$. Appendix H.3 also introduces concrete trimming and kernel functions to satisfy our Assumption $3'$. Furthermore, Appendix G presents conditions under which the proposed implementation procedure is compatible with our Assumptions $1'$ and $2'$.

## 4 Simulation Studies

In this section, we conduct simulation studies for inference of average treatment effects based on the inverse propensity score weighting discussed in Section 3.2.

We generate data $X = (Y, D, W)$ by the following procedure. The observed outcome $Y$ is given by $Y = (1-D)Y_0 + DY_1$, where $D$ is the treatment selection indicator, $Y_0$ is the potential outcome under no treatment, and $Y_1$ is the potential outcome under treatment. The potential outcomes are generated by $Y_d = \mathbb{1}\{d=1\} + W^T\beta_d + U_d$ where $U_d \sim N(0, 0.5^2)$ for each treatment status $d \in \{0,1\}$. The covariates $W$ of dimension $\dim(W)$ are generated by $W \sim N(0, I_{\dim(W)})$, where $I_{\dim(W)}$ is the $\dim(W) \times \dim(W)$ identity matrix. The treatment selection rule is $D = \mathbb{1}\left\{W^T\gamma \geq \varepsilon\right\}$ where $\varepsilon \sim$ standard logistic distribution. The parameter vectors are set to $\gamma = c_\gamma\sqrt{\frac{2}{\dim(W)+\dim(W)^2}}\left(1, \sqrt{2}, ..., \sqrt{\dim(W)}\right)^T$ and $\beta_d = c_{\beta_d}\sqrt{\frac{2}{\dim(W)+\dim(W)^2}}\left(1, \sqrt{2}, ..., \sqrt{\dim(W)}\right)^T$ for each $d \in \{0,1\}$. These definitions are made so that we can conveniently control the scales as $\|\gamma\| = |c_\gamma|$ and $\|\beta_d\| = |c_{\beta_d}|$ for each $d \in \{0,1\}$. Note that the average treatment effect is $\theta = E\left[Y_1 - Y_0\right] = 1$ in this setup. Across sets of simulations, we vary the data generating parameters $n$, $\dim(W)$, $c_\gamma$, and $\beta_d$. Each set of simulations consists of 10,000 iterations.

For regularized estimation, we use the trimming function $S$ defined by $S(u) = \mathbb{1}\{0 \leq u \leq 1\} \cdot (6u^5 - 15u^4 + 10u^3) + \mathbb{1}\{1 < u\}$. For bias correction, we use the one-sided quinque-weight kernel function $K$ defined by $K(u) \propto \mathbb{1}\{0 \leq u \leq 1\} \cdot (1-u^2)^5$. These choices satisfy Assumption $3'$ on trimming and kernel functions. The bandwidth choice, estimation, and inference procedures precisely follow the steps in the guide for practice outlined in Section H.3. For comparison with benchmarks, we present simulation results for each of the following four methods: (I) our trimmed estimator with optimal bandwidth and bias correction (exactly based on the guide in Section H.3); (II) the trimmed estimator with optimal bandwidth without bias correction; (III) the trimmed estimator with rule-of-thumb bandwidth, specifically $h = 0.1$ (cf. Crump, Hotz, Imbens and Mitnik, 2009); and (IV) the untrimmed estimator, i.e., $h = 0.0$.

Table 1 displays simulation results of the root mean square error (RMSE) and coverage frequency by the estimated 95% confidence interval (95% Coverage). Columns (I)–(IV) indicate the four methods listed above. Note that, as the scale $c_\gamma$ of the logit parameters increases, we tend to have a larger frequency of observations in a sample with the propensity scores $\pi(W, \gamma)$ close to zero and one, and thus the denominator $A(\gamma)$ close to zero. In other words, rows with larger values of $c_\gamma$ are associated with greater adversity for inference. The simulation results indeed evidence this feature: both the RMSE and the 95% coverage frequencies become worse as $c_\gamma$ increases. Nonetheless, the four methods (I)–(IV) exhibit different sensitivities to the increase in $c_\gamma$. We observe the following two points.

First, the simulated RMSE are worse in column (IV) than for those in any of the columns (I), (II) and (III). Recall that (I), (II) and (III) use trimming, whereas (IV) does not. This result is consistent with the theory that trimming improves the convergence rate and thus the approximate RMSE in finite sample. The RMSE under (III) are smaller than those under (II) as the former method trim observations more aggressively than the latter especially for larger sample sizes. However, this advantage of (III) is achieved at the expense of larger biases resulting in valid inference. The RMSE under (I) are larger than those under (II) as the former entails additional variances from bias estimation for the sake of valid inference.

Second, the simulated 95% coverage frequencies in column (I) are closer to the nominal probability 0.950 than those in any of the columns (II), (III) and (IV). The observation that (I) yields better results than (II) or (III) is consistent with the theory that (I) entails asymptotically valid inference whereas (II) or (III) does not. The observation that (I) also yields slightly better results than (IV) may be associated with the better convergence rates for the former than the latter due to trimming. In summary, simulation results evidence that our trimmed estimator with optimal bandwidth and bias correction provides the most accurate inference outcomes.

# 5 Summary

This paper proposes a new method of inference for moments of ratios of the form $E[B/A]$. Our assumptions are simple, easily verifiable with concrete function spaces, and general enough to encompass a wide spectrum of models including those cases where a naïve sample mean converges slowly and a trimming improves the convergence rate. The main purpose of this method is to deal with a number of practical concerns in a theoretically coherent way. Our method provides the following two practical values.

First, our bias correction framework allows for a use of larger trimming thresholds, e.g., $h \propto n^{-1/5}$ in the baseline case, which admit faster convergence rates. This feature proves useful in practice, as evidenced in our simulation studies (Section 4). Furthermore, our bias correction approach admits a systematic method of choosing trimming thresholds in practice. By balancing the variance and the bias of a unit order less than the order of supposed smoothness, we obtain the data-driven trimming rule that is consistent with valid inference while achieving a close-to-optimal convergence rate.

Second, the rate-adaptive method of inference based on the self-normalized sum allows for valid inference without a prior knowledge about the unknown convergence rate. This feature is useful as it eliminates the need for a practitioner to pre-estimate a parameter that determines the convergence rate, such as the curvature parameter of the density function in a neighborhood of $A$. As such, the practical procedure that we propose (Section H) indeed consists of very simple steps, and its computational cost is actually minimal.

In summary, the combination of the rate-adaptive method and the trimming bias correction accounting for the asymptotic variance of the bias estimator as well allows for the twofold robustness in inference for the moment $E[B/A]$, namely against large trimming bias and unknown convergence rate. We believe that this paper will contribute to empirical practice by providing this robustly valid inference procedure with the user-friendly and data-driven implementation procedure.

# Appendix

## A Proofs of the Main Results

### A.1 Proof of Lemma 2.1

*Proof.* Under Assumptions 1 (i) and 2 (i)–(ii), we can write

$$\theta - \theta_h = \int_0^h \frac{f_A(a) \cdot E[B|A=a]}{a} da = \int_0^h \frac{\tau_1(a) - \tau_1(0)}{a} da$$

where the first equality is due to the law of iterated expectations, and the second equality follows from Assumption 2 (ii). By the $k$-order mean value expansion under Assumption 2 (ii), we can write the difference in the last expression as

$$\tau_1(a) - \tau_1(0) = \sum_{\kappa=1}^{k-1} \frac{a^\kappa}{\kappa!} \tau_1^{(\kappa)}(0) + a^k \cdot R^k(\alpha^k(a))$$

with $\alpha^k(a) \in (0, a)$, where the remainder function $R^k$ given by $R^k(a) = \frac{1}{k!}\tau_1^{(k)}(a)$ is uniformly bounded in absolute value on $[0, \eta]$ for some small $\eta > 0$ by Assumption 2 (ii). Combining the above results together, we can now write

$$\begin{aligned} \theta - \theta_h &= \sum_{\kappa=1}^{k-1} \frac{1}{\kappa!} \tau_1^{(\kappa)}(0) \int_0^h a^{\kappa-1} da + \int_0^h a^{k-1} R^k\left(\alpha^k(a)\right) da \\ &= \sum_{\kappa=1}^{k-1} \frac{1}{\kappa!} \frac{1}{\kappa} h^\kappa \tau_1^{(\kappa)}(0) + \int_0^h a^{k-1} R^k\left(\alpha^k(a)\right) da \end{aligned}$$

where the remaining bias is bounded in absolute value as

$$\left| \int_0^h a^{k-1} R^k\left(\alpha^k(a)\right) da \right| \leq \int_0^h a^{k-1} da \sup_{a \in (0,h)} \left| R^k\left(\alpha^k(a)\right) \right| = \frac{1}{k} h^k \sup_{a \in (0,h)} \left| R^k\left(\alpha^k(a)\right) \right|.$$

Finally, noting that $\sup_{a \in [0,\eta]} \left| R^k(a) \right| < \infty$ proves the claim for $h \leq \eta$. □

### A.2 Proof of Theorem 2.1

*Proof.* First, by the definition of $\widehat{\tau}_1^{(\kappa)}(0)$ given in (2.6), we can write

$$-E[\widehat{P}_h^{k-1}] = E\left[\sum_{\kappa=1}^{k-1} \frac{\rho_\kappa h^\kappa}{\kappa} \widehat{\tau}_1^{(\kappa)}(0)\right] = \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa h^\kappa}{\kappa} E\left[\widehat{\tau}_1^{(\kappa)}(0)\right] = \sum_{\kappa=1}^{k-1} \frac{(-1)^\kappa}{\kappa h} \rho_\kappa E\left[BK^{(\kappa)}\left(\frac{A}{h}\right)\right]. \tag{A.1}$$

By Assumptions 2 (i) and 3 together with the definition of $\tau_1$ given in (2.4), the last expression in (A.1) may be rewritten as

$$\sum_{\kappa=1}^{k-1}\frac{(-1)^{\kappa}}{\kappa h}\rho_{\kappa}E\left[BK^{(\kappa)}\left(\frac{A}{h}\right)\right]=\sum_{\kappa=1}^{k-1}\frac{(-1)^{\kappa}}{\kappa h}\rho_{\kappa}\int_0^h \tau_1(a)K^{(\kappa)}\left(\frac{a}{h}\right)da \tag{A.2}$$

From the proof of Lemma 2.1, $\tau_1(a)=\tau_1(a)-\tau_1(0)=\sum_{\kappa=1}^{k-1}\frac{a^{\kappa}}{\kappa!}\tau_1^{(\kappa)}(0)+a^k\cdot R^k(\alpha^k(a))$ with $\alpha^k(a)\in(0,a)$, where the remainder function $R^k$ given by $R^k(a)=\frac{1}{k!}\tau_1^{(k)}(a)$ is uniformly bounded in absolute value on $[0,\eta]$ for some small $\eta>0$ by Assumption 2 (ii). Substituting this mean value expansion in the last expression in (A.2) yields

$$\begin{aligned}
&\sum_{\kappa=1}^{k-1}\frac{(-1)^{\kappa}}{\kappa h}\rho_{\kappa}\int_0^h \tau_1(a)K^{(\kappa)}\left(\frac{a}{h}\right)da\\
&=\sum_{\kappa_1=1}^{k-1}\frac{1}{\kappa_1!}\frac{1}{\kappa_1}h^{\kappa_1}\tau_1^{(\kappa_1)}(0)\kappa_1\sum_{\kappa=1}^{k-1}\rho_{\kappa}\frac{(-1)^{\kappa}}{\kappa}\int_0^1 u^{\kappa_1}K^{(\kappa)}(u)\,du && \text{(A.3)}\\
&\quad+h^k\sum_{\kappa=1}^{k-1}\rho_{\kappa}\frac{(-1)^{\kappa}}{\kappa}\int_0^1 R^k(\alpha^k(uh))u^kK^{(\kappa)}(u)\,du, && \text{(A.4)}
\end{aligned}$$

where the last equality is due to changes of variables. The expression in line (A.3) reduces to

$$\sum_{\kappa_1=1}^{k-1}\frac{1}{\kappa_1!}\frac{1}{\kappa_1}h^{\kappa_1}\tau_1^{(\kappa_1)}(0)\kappa_1\sum_{\kappa=1}^{k-1}\rho_{\kappa}\frac{(-1)^{\kappa}}{\kappa}\int_0^1 u^{\kappa_1}K^{(\kappa)}(u)\,du=-\widehat{P}_h^{k-1} \tag{A.5}$$

by the definition of $\widehat{P}_h^{k-1}$ and the choice of $\{\rho_\kappa\}_{\kappa=1}^{k-1}$ to satisfy (2.8). To see the asymptotic behavior of line (A.4), note that

$$\left|\sum_{\kappa=1}^{k-1}\rho_{\kappa}\frac{(-1)^{\kappa}}{\kappa}\int_0^1 R^k(\alpha^k(uh))u^kK^{(\kappa)}(u)\,du\right|\le\sum_{\kappa=1}^{k-1}\frac{|\rho_{\kappa}|}{\kappa}\int_0^1 u^k\left|K^{(\kappa)}(u)\right|du\sup_{a\in(0,h)}\left|R^k\left(\alpha^k(a)\right)\right|,$$

where $\int_0^1 u^k\left|K^{(\kappa)}(u)\right|du<\infty$ for each $\kappa\in\{1,...,k-1\}$ by Assumption 3, and $h\mapsto\sup_{a\in(0,h)}\left|R^k\left(\alpha^k(a)\right)\right|$ is uniformly bounded on $[0,\eta]$. Therefore,

$$h^k\sum_{\kappa=1}^{k-1}\rho_{\kappa}\frac{(-1)^{\kappa}}{\kappa}\int_0^1 R^k(\alpha^k(uh))u^kK^{(\kappa)}(u)\,du=O(h^k) \tag{A.6}$$

as $h\to 0$. Combining the chains of equalities from (A.1)–(A.6), we obtain

$$E[\widehat{P}_h^{k-1}]=P_h^{k-1}+O(h^k) \tag{A.7}$$

as $h \to 0$. On the other hand, from Lemma 2.1, we also have

$$\theta_h - \theta = P_h^{k-1} + O(h^k) \tag{A.8}$$

as $h \to 0$. Combining (A.7) and (A.8) yields $E\left[\widehat{\theta}_h\right] - E[\widehat{P}_h^{k-1}] = \theta + O(h^k)$ as $h \to 0$. □

## A.3 Proof of Lemma 2.2

*Proof.* First, we obtain

$$Var\left(\frac{E_n[(Z_n - E[Z_n])^2]}{E[(Z_n - E[Z_n])^2]}\right) = \frac{1}{n} Var\left(\frac{(Z_n - E[Z_n])^2}{E[(Z_n - E[Z_n])^2]}\right) = \frac{1}{n}\left(\frac{E[(Z_n - E[Z_n])^4]}{E[(Z_n - E[Z_n])^2]^2} - 1\right) = o(1)$$

as $n \to \infty$ and $nh_n^2 \to \infty$ by i.i.d. sampling and Lemma C.1 (i) under Assumptions 1, 2 (iii), and 3. Therefore, using Chebyshev's inequality yields

$$\frac{E_n[(Z_n - E[Z_n])^2]}{E[(Z_n - E[Z_n])^2]} \to_p 1 \qquad \text{as } n \to \infty. \tag{A.9}$$

Second, note that Lindeberg condition holds for the triangular array

$$\left\{ (n,i) \mapsto n^{-1/2} \frac{Z_{n,i} - E[Z_n]}{E[(Z_n - E[Z_n])^2]^{1/2}} \middle| i \in \{1, ..., n\}, n \in \mathbb{N} \right\}$$

because Lyapunov condition,

$$n^{-\delta/2} \frac{E\left[|Z_n - E[Z_n]|^{2+\delta}\right]}{E\left[(Z_n - E[Z_n])^2\right]^{(2+\delta)/2}} \to 0$$

as $n \to \infty$ and $nh_n^2 \to \infty$, is satisfied in particular with $\delta = 2$ by Lemma C.1 under Assumptions 1, 2 (iii), and 3. Therefore, applying Lindeberg-Feller Theorem yields

$$n^{1/2} \frac{E_n[Z_n] - E[Z_n]}{E[(Z_n - E[Z_n])^2]^{1/2}} \to_d N(0,1) \qquad \text{as } n \to \infty. \tag{A.10}$$

Third, applying Continuous Mapping Theorem and Slutsky's Theorem to (A.9) and (A.10), we obtain

$$n^{1/2} \frac{E_n[Z_n] - E[Z_n]}{E_n[(Z_n - E[Z_n])^2]^{1/2}} \to_d N(0,1)$$

as $n \to \infty$. Finally, using the generic identical equality

$$P\left(n^{1/2} \frac{E_n[Z_n] - E[Z_n]}{E_n[(Z_n - E_n[Z_n])^2]^{1/2}} \geq x\right) = P\left(n^{1/2} \frac{E_n[Z_n] - E[Z_n]}{E_n[(Z_n - E[Z_n])^2]^{1/2}} \geq x \cdot \left(1 + \frac{x^2}{n}\right)^{-1/2}\right),$$

we obtain the desired result. □

### A.4 Proof of Theorem 2.2

*Proof.* First, consider the ratio

$$\frac{\widehat{\sigma}_n^2}{\sigma_n^2} - 1 = \frac{E_n\left[Z_n^2 - E[Z_n^2]\right]}{\sigma_n^2} - \frac{\widehat{\theta}_{h_n}^2 - \theta_{h_n}^2}{\sigma_n^2} \tag{A.11}$$

The first term on the right-hand side of (A.11) has the mean and the variance

$$E\left[\frac{E_n\left[Z_n^2 - E[Z_n^2]\right]}{\sigma_n^2}\right] = 0$$

$$Var\left(\frac{E_n\left[Z_n^2 - E[Z_n^2]\right]}{\sigma_n^2}\right) = n^{-1}\frac{E[Z_n^4 - E[Z_n^2]^2]}{E[(Z_n - E[Z_n])^2]^2} \to 0$$

as $n \to \infty$ and $nh_n^2 \to \infty$ by Lemma C.1 (ii) under Assumptions 1, 2 (iii), and 3. Therefore, by the weak law of large numbers for triangular array, we obtain

$$\frac{E_n\left[Z_n^2 - E[Z_n^2]\right]}{\sigma_n^2} = o_p(1)$$

as $n \to \infty$. The second term on the right-hand side of (A.11) has the numerator

$$\widehat{\theta}_{h_n}^2 - \theta_{h_n}^2 = o_p(1)$$

as $n \to \infty$ and $nh_n^2 \to \infty$ by Lemma C.2 under Assumption 1 (i)–(ii). Therefore, (A.11) is

$$\frac{\widehat{\sigma}_n^2}{\sigma_n^2} - 1 = o_p(1) \tag{A.12}$$

as $n \to \infty$ and $nh_n^2 \to \infty$ by Lemma C.3 under Assumptions 1 (i) and 1 (iii). Now, combining (A.12), Lemma C.3 under Assumptions 1 (i) and 1 (iii), and Theorem 2.1 under Assumptions 1 (i), 2 (i)–(ii), and 3 together imply

$$\begin{aligned} n^{1/2}\widehat{\sigma}_n^{-1}\left(E\left[\widehat{\theta}_{h_n}\right] - E\left[\widehat{P}_{h_n}^{k-1}\right] - \theta\right) &= n^{1/2}\sigma_n^{-1}\left(\frac{\widehat{\sigma}_n^2}{\sigma_n^2}\right)^{-1/2}\left(E\left[\widehat{\theta}_{h_n}\right] - E\left[\widehat{P}_{h_n}^{k-1}\right] - \theta\right) \\ &= n^{1/2}\sigma_n^{-1}\left(1 + o_p(1)\right)^{-1/2} O(h_n^k) = o_p(1) \end{aligned} \tag{A.13}$$

as $n \to \infty$ and $nh_n^{2k} \to 0$. Finally, combining this equation (A.13) and Lemma 2.2 under Assumptions 1, 2 (iii), and 3 yields

$$\begin{aligned} n^{1/2}\widehat{\sigma}_n^{-1}\left(\widehat{\theta}_h - \widehat{P}_h^{k-1} - \theta\right) =& n^{1/2}\widehat{\sigma}_n^{-1}\left(E_n[Z_n] - E[Z_n]\right) \\ &+ n^{1/2}\widehat{\sigma}_n^{-1}\left(E\left[\widehat{\theta}_{h_n}\right] - E\left[\widehat{P}_{h_n}^{k-1}\right] - \theta\right) \to_d N(0,1) \end{aligned}$$

as $n \to \infty$ and $nh_n^{2k} \to 0$ □

## B Additional Proof: Theorem 2.3

*Proof.* Note that $\widehat{\theta}_{h_n}$ and $\widehat{P}_{h_n}^{k-1}$ are independent under Assumption 3. From (2.6) and (2.7), it suffices to show that

$$Var\left(\rho_\kappa h_h^\kappa \widehat{\tau}_1^{(\kappa)}(0)\right)/Var\left(\widehat{\theta}_{h_n}\right) = O(1)$$

as $n \to \infty$. By the i.i.d. sampling, we first rewrite the denominator and the numerator as

$$Var\left(\widehat{\theta}_{h_n}\right) = \frac{1}{n}Var\left(\frac{B}{A}\cdot \mathbb{1}\{A > h_n\}\right) = \frac{1}{n}\left(E\left[\frac{B^2}{A^2}\cdot \mathbb{1}\{A > h_n\}\right] - E\left[\frac{B}{A}\cdot \mathbb{1}\{A > h_n\}\right]^2\right)$$

and

$$\begin{aligned} Var\left(\rho_\kappa h^\kappa \widehat{\tau}_1^{(\kappa)}(0)\right) &= \rho_\kappa^2 \frac{1}{\kappa^2}\frac{1}{nh_n^2} Var\left(BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right) \\ &= \rho_\kappa^2 \frac{1}{\kappa^2}\frac{1}{nh_n^2}\left(E\left[B^2 K^{(\kappa)}\left(\frac{A}{h_n}\right)^2\right] - E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right]^2\right). \end{aligned}$$

Therefore, we obtain

$$Var\left(\rho_\kappa h_n^\kappa \widehat{\tau}_1^{(\kappa)}(0)\right)/Var\left(\widehat{\theta}_{h_n}\right) = c\cdot h_n^{-2}\cdot \frac{E\left[B^2K^{(\kappa)}\left(\frac{A}{h_n}\right)^2\right] - E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right]^2}{E\left[\frac{B^2}{A^2}\cdot \mathbb{1}\{A > h_n\}\right] - E\left[\frac{B}{A}\cdot \mathbb{1}\{A > h_n\}\right]^2}, \tag{B.1}$$

where $c = \rho_\kappa^2/\kappa^2 \in (0,\infty)$. To analyze the asymptotic order of the the right-hand side of the above equation, we now branch into two cases.

**Case 1:** $\tau_2(0) = 0$.

$$\frac{1}{h_n}E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right] = \frac{1}{h_n}\int_0^{h_n}\tau_1(a)K^{(\kappa)}\left(\frac{a}{h_n}\right)da = \int_0^1 \tau_1(uh_n)K^{(\kappa)}(u)\,du = O(1)$$

as $n \to \infty$ under Assumptions 1 (i), 2 (i)–(ii) and 3. From Assumption 2 (iii), $\tau_2(a) = \tau_2(0) + a\cdot R_2^1(\alpha_2^1(a))$ with $\alpha_2^1(a) \in (0,a)$, where the remainder function $R_2^1$ given by $R_2^1(a) = \tau_2'(a)$ is uniformly bounded in absolute value on $[0,\eta]$ for some small $\eta > 0$. Therefore, using $\tau_2(0) = 0$ yields

$$\begin{aligned} \frac{1}{h_n^2}E\left[B^2K^{(\kappa)}\left(\frac{A}{h_n}\right)^2\right] &= \frac{1}{h_n^2}\int_0^{h_n}\tau_2(a)K^{(\kappa)}\left(\frac{a}{h_n}\right)^2 da = \frac{1}{h_n}\int_0^1 \tau_2(uh_n)K^{(\kappa)}(u)^2\,du \\ &= \int_0^1 u\cdot R_2^1(\alpha_2^1(uh_n))K^{(\kappa)}(u)^2\,du = O(1) \end{aligned}$$

as $n \to \infty$ under Assumptions 1 (i), 2 (i) and 3. This shows that

$$h_n^{-2}\left(E\left[B^2 K^{(\kappa)}\left(\frac{A}{h_n}\right)^2\right] - E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right]^2\right) = O(1)$$

as $n \to \infty$. By Lemma C.3 under Assumptions 1 (i) and 1 (iii), therefore, the variance ratio (B.1) is $O(1)$ as $n \to \infty$.

**Case 2:** $\tau_2(0) > 0$. Since Assumptions 1 (i), 2 (i)–(ii) and 3 yield

$$\frac{1}{h_n} E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right] = O(1)$$

as $n \to \infty$, as argued above, we have

$$\frac{1}{h_n} E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right]^2 = o(1)$$

Furthermore, Assumption 2 (iii) provides

$$\begin{aligned}\frac{1}{h_n} E\left[B^2 K^{(\kappa)}\left(\frac{A}{h_n}\right)^2\right] &= \frac{1}{h_n}\int_0^{h_n} \tau_2(a) K^{(\kappa)}\left(\frac{a}{h_n}\right)^2 da = \int_0^1 \tau_2(uh_n) K^{(\kappa)}(u)^2\, du \\ &= \tau_2(0)\int_0^1 K^{(\kappa)}(u)^2\, du + h_n \int_0^1 u \cdot R_2^1(\alpha_2^1(uh_n)) K^{(\kappa)}(u)^2\, du = O(1)\end{aligned}$$

as $n \to \infty$ under Assumptions 1 (i), 2 (i), and 3. Combining the above two equations, we obtain

$$\frac{1}{h_n} E\left[B^2 K^{(\kappa)}\left(\frac{A}{h_n}\right)^2\right] - \frac{1}{h_n} E\left[BK^{(\kappa)}\left(\frac{A}{h_n}\right)\right]^2 = O(1) \tag{B.2}$$

as $n \to \infty$. Note that Assumption 1 (i) yields

$$h_n E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h_n\}\right]^2 = o(1)$$

as $n \to \infty$. Assumption 2 (iii) provides

$$\begin{aligned} h_n E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h_n\}\right] &= h_n \int_{h_n}^{\infty} \tau_2(a) \frac{1}{a^2} da = \int_1^{\infty} \tau_2(uh_n)\frac{1}{u^2} du \geq \int_1^2 \tau_2(uh_n)\frac{1}{u^2} du \\ &\geq \int_1^2 \tau_2(uh_n)\frac{1}{4} du = \frac{1}{4}\int_1^2 \tau_2(uh_n) du = \frac{1}{4}\int_1^2 (\tau_2(0) + uh_n R_2^1(\alpha_2^1(uh_n)) du \\ &= \frac{1}{4}\tau_2(0) + h_n \frac{1}{4}\int_1^2 u R_2^1(\alpha_2^1(uh_n)) du = \frac{1}{4}\tau_2(0) + o(1)\end{aligned}$$

as $n \to \infty$. Combining the above two equations, we conclude that

$$h_n E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h_n\}\right] - h_n E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h_n\}\right]^2 = \frac{1}{4}\tau_2(0) + o(1) \geq \frac{1}{8}\tau_2(0) > 0 \tag{B.3}$$

for all $h_n$ as $n \to \infty$. By (B.2) and (B.3), therefore, the variance ratio (B.1) is $O(1)$ as $n \to \infty$. □

# C Auxiliary Lemmas for the Main Results

## C.1 Auxiliary Lemma: $L^4$ to $L^2$ Ratio

**Lemma C.1** ($L^4$ to $L^2$ Ratio)**.** *If Assumptions 1, 2 (iii), and 3 are satisfied, then*

$$\text{(i) } n^{-1/4} \cdot \frac{E[(Z_n - E[Z_n])^4]^{1/4}}{E[(Z_n - E[Z_n])^2]^{1/2}} \to 0 \qquad \text{and} \qquad \text{(ii) } n^{-1/4} \cdot \frac{E[(Z_n^4 - E[Z_n^2]^2)]^{1/4}}{E[(Z_n - E[Z_n])^2]^{1/2}} \to 0$$

*as* $nh_n^2 \to \infty$.

*Proof.* (i) Since $Z_n = \frac{B}{A} \cdot \mathbb{1}\{A > h\} + \sum_{\kappa=1}^{k-1} \rho_\kappa \frac{1}{\kappa} h^{-1} (-1)^\kappa B K^{(\kappa)} \left(\frac{A}{h}\right)$, Minkowski's inequality yields

$$E[(Z_n - E[Z_n])^4]^{1/4} \leq E\left[\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\} - E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]\right)^4\right]^{1/4} + \sum_{\kappa=1}^{k-1} \rho_\kappa \frac{1}{\kappa} h^{-1} E\left[\left(BK^{(\kappa)}\left(\frac{A}{h}\right) - E\left[BK^{(\kappa)}\left(\frac{A}{h}\right)\right]\right)^4\right]^{1/4}. \tag{C.1}$$

Furthermore, Assumption 3 and i.i.d. sampling imply that

$$\frac{B}{A} \cdot \mathbb{1}\{A > h\} \qquad \text{and} \qquad \sum_{\kappa=1}^{k-1} \rho_\kappa \frac{1}{\kappa} h^{-1} (-1)^\kappa B K^{(\kappa)} \left(\frac{A}{h}\right)$$

are independent, and hence

$$E[(Z_n - E[Z_n])^2] \geq E\left[\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\} - E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]\right)^2\right]. \tag{C.2}$$

Since $K^{(\kappa)}$ is bounded under Assumption 3 and $E[B^4] < \infty$ under Assumption 1 (ii), we have

$$h^{-1} E\left[\left(BK^{(\kappa)}\left(\frac{A}{h}\right) - E\left[BK^{(\kappa)}\left(\frac{A}{h}\right)\right]\right)^4\right] = O(h^{-1}) \tag{C.3}$$

for each $\kappa \in \{1, ..., k-1\}$. By (C.1), (C.2), (C.3), $nh \to \infty$, and Lemma C.3 under Assumptions 1 (i) and 1 (iii), it suffices to show that

$$\frac{E\left[\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\} - E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]\right)^4\right]}{nE\left[\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\} - E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]\right)^2\right]^2} = o(1).$$

We branch into two cases below. Throughout, we use the property that $\tau_j(a) = \tau_j(0) + R_j^1(\alpha_j^1(a))$ with $\alpha_j^1(a) \in (0, a)$, where the remainder function $R_j^1$ given by $R_j^1(a) = \tau_j'(a)$ is uniformly bounded in absolute value on $[0,\eta]$ for some small $\eta > 0$ for each $j \in \{2,3,4\}$ under Assumption 2 (iii).

**Case 1:** $\tau_2(0) = 0$. By $nh^2 \to \infty$ and Lemma C.3 under Assumptions 1 (i) and 1 (iii), it suffices to show that

$$E\left[\left(\frac{B}{A}\cdot \mathbb{1}\{A > h\} - E\left[\frac{B}{A}\cdot \mathbb{1}\{A > h\}\right]\right)^4\right] = O(h^{-2}) \tag{C.4}$$

as $h \to 0$. The following four parts together show (C.4). First,

$$E\left[\frac{B}{A}\cdot \mathbb{1}\{A > h\}\right] = O(1)$$

as $h \to 0$ under Assumption 1 (i). Second, for $h \in (0, \eta^2)$, we can write

$$\begin{aligned}
E\left[\frac{B^2}{A^2}\cdot \mathbb{1}\{A > h\}\right] &= E\left[\frac{B^2}{A^2}\cdot \mathbb{1}\{h < A \le h^{1/2}\}\right] + E\left[\frac{B^2}{A^2}\cdot \mathbb{1}\{A > h^{1/2}\}\right]\\
&\le \int_h^{h^{1/2}} \frac{\tau_2(a)}{a^2} da + h^{-1}E\left[B^2\cdot \mathbb{1}\{A > h^{1/2}\}\right]\\
&= \int_h^{h^{1/2}} \frac{R_2^1(\alpha_2^1(a))}{a} da + h^{-1}E\left[B^2\cdot \mathbb{1}\{A > h^{1/2}\}\right]\\
&= \int_1^{h^{-1/2}} \frac{R_2^1(\alpha_2^1(uh))}{u} du + h^{-1}E\left[B^2\cdot \mathbb{1}\{A > h^{1/2}\}\right]\\
&= O(\log(h^{-1/2})) + O(h^{-1}) = O(h^{-1})
\end{aligned}$$

as $h \to 0$ under Assumption 1 (ii) and $\tau_2(0) = 0$. Third, by similar lines of calculations for $h \in (0, \eta^2)$, we have

$$\begin{aligned}
E\left[\frac{B^3}{A^3}\cdot \mathbb{1}\{A > h\}\right] &= E\left[\frac{B^3}{A^3}\cdot \mathbb{1}\{h \le A \le h^{1/2}\}\right] + E\left[\frac{B^3}{A^3}\cdot \mathbb{1}\{A > h^{1/2}\}\right]\\
&= O(h^{-1/2}) + O(h^{-3/2}) = O(h^{-3/2})
\end{aligned}$$

as $h \to 0$ under Assumption 1 (ii). Fourth, by similar lines of calculations for $h \in (0, \eta^2)$ again, we have

$$\begin{aligned}
E\left[\frac{B^4}{A^4}\cdot \mathbb{1}\{A > h\}\right] &= E\left[\frac{B^4}{A^4}\cdot \mathbb{1}\{h \le A \le h^{1/2}\}\right] + E\left[\frac{B^4}{A^4}\cdot \mathbb{1}\{A > h^{1/2}\}\right]\\
&= O(h^{-1}) + O(h^{-2}) = O(h^{-2})
\end{aligned}$$

as $h \to 0$ under Assumption 1 (ii). Therefore, (C.4) holds.

**Case 2**: $\tau_2(0) \ne 0$. Since we let $nh^2 \to \infty$, it suffices to show

(i) that $hE\left[\left(\frac{B}{A}\cdot\mathbb{1}\{A>h\}-E\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]\right)^2\right]$ is bounded away from zero in a neighborhood of $h=0$; and

(ii) that $E\left[\left(\frac{B}{A}\cdot\mathbb{1}\{A>h\}-E\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]\right)^4\right]=O(h^{-3})$.

First, to show (i), we make the following lines of calculations.

$$\begin{aligned}
&hE\left[\left(\frac{B}{A}\cdot\mathbb{1}\{A>h\}-E\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]\right)^2\right]\\
&=hE\left[\frac{B^2}{A^2}\cdot\mathbb{1}\{A>h\}\right]-hE\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]^2\\
&=h\int_h^\infty\frac{\tau_2(a)}{a^2}da-hE\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]^2\\
&\geq h\int_h^{2h}\frac{\tau_2(a)}{a^2}da-hE\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]^2\\
&=\int_1^2\frac{\tau_2(uh)}{u^2}du-hE\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]^2\\
&=\int_1^2\frac{\tau_2(0)+uhR_2^1(\alpha_2^1(uh))}{u^2}du-hE\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]^2\\
&\geq\frac{1}{2}\tau_2(0)-h\left(\frac{1}{2}\int_1^2\left|R_2^1(\alpha_2^1(uh))\right|du-E\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]^2\right).
\end{aligned}$$

The last expression is bounded away from zero in a neighborhood of $h=0$ by $\tau_2(0)\neq 0$ and Assumptions 1 (i) and 2 (iii). Second, to show (ii), we note that

$$E\left[\frac{B}{A}\cdot\mathbb{1}\{A>h\}\right]=O(1),\qquad E\left[\frac{B^2}{A^2}\cdot\mathbb{1}\{A>h\}\right]=O(h^{-2}),$$
$$\text{and}\qquad E\left[\frac{B^3}{A^3}\cdot\mathbb{1}\{A>h\}\right]=O(h^{-3})$$

as $h\to 0$ under Assumption 1 (ii). For the fourth moment, similar lines of calculations to those in case 1 yield

$$\begin{aligned}
E\left[\frac{B^4}{A^4}\cdot\mathbb{1}\{A>h\}\right]&=\int_h^{h^{1/2}}\frac{\tau_4(a)}{a^4}da+E\left[\frac{B^4}{A^4}\cdot\mathbb{1}\{A>h^{1/2}\}\right]\\
&=\int_h^{h^{1/2}}\frac{\tau_4(0)+aR_4^1(\alpha_4^1(a))}{a^4}da+E\left[\frac{B^4}{A^4}\cdot\mathbb{1}\{A>h^{1/2}\}\right]\\
&=\int_h^{h^{1/2}}\frac{\tau_4(0)}{a^4}da+\int_h^{h^{1/2}}\frac{R_4^1(\alpha_4^1(a))}{a^3}da+E\left[\frac{B^4}{A^4}\cdot\mathbb{1}\{A>h^{1/2}\}\right]\\
&=O(h^{-3})+O(h^{-2})+O(h^{-2})=O(h^{-3})
\end{aligned}$$

as $h \to 0$ under Assumption 1 (ii). This completes a proof of part (i). A proof of part (ii) similarly follows. $\square$

## C.2 Auxiliary Lemma: $\widehat{\theta}_{h_n} - \theta_{h_n} \to_p 0$

**Lemma C.2.** *If Assumption 1 (i)–(ii) is satisfied, then* $\widehat{\theta}_{h_n} - \theta_{h_n} \to_p 0$ *as* $n \to \infty$ *and* $nh_n^2 \to \infty$.

*Proof.* First, note that $E[\widehat{\theta}_{h_n}] = \theta_{h_n}$ holds by the definitions of $\widehat{\theta}_{h_n}$ and $\theta_{h_n}$. On the other hand, the variance can be written as

$$Var\left(\widehat{\theta}_{h_n}\right) = n^{-1}E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h_n\}\right] - n^{-1}E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h_n\}\right]^2. \tag{C.5}$$

under the i.i.d. sampling. The first term on the right-hand side of (C.5) is

$$n^{-1}E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h_n\}\right] \leq n^{-1}h^{-2}E\left[B^2 \cdot \mathbb{1}\{A > h_n\}\right] = O(n^{-1}h^{-2})$$

under Assumption 1 (ii). The first term on the right-hand side of (C.5) is

$$n^{-1}E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h_n\}\right] = O(n^{-1})$$

under Assumption 1 (i). This shows that $Var\left(\widehat{\theta}_{h_n}\right)$ in (C.5) is $o(1)$ as $n \to \infty$ and $nh_n^2 \to \infty$. Therefore, by the weak law of large numbers for triangular array, we have $\widehat{\theta}_{h_n} - \theta_{h_n} \to_p 0$ as $n \to \infty$ and $nh_n^2 \to \infty$. $\square$

## C.3 Auxiliary Lemma: Positive Variance

**Lemma C.3.** *If Assumptions 1 (i) and 1 (iii) are satisfied, then there exists* $\epsilon > 0$ *such that*

$$Var\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right) > \epsilon$$

*holds for all* $h$ *in a neighborhood of* $0$.

*Proof.* Since $\frac{B}{A}$ is integrable under Assumption 1, an application of the dominated convergence theorem yields

$$E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right] \to E\left[\frac{B}{A}\right]$$

as $h \to 0$. This implies that for each $\epsilon' > 0$ there exist $h_{\epsilon',1} > 0$ such that

$$E\left[\frac{B}{A}\right]^2 \geq E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]^2 - \frac{\epsilon'}{3} \tag{C.6}$$

for all $h < h_{\epsilon',1}$. Furthermore, since $E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h\}\right]$ is non-increasing in $h$, an application of the monotone convergence theorem gives

$$E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h\}\right] \to E\left[\frac{B^2}{A^2}\right]$$

as $h \to 0$. This implies that for each $\epsilon' > 0$ there exist $h_{\epsilon',2} > 0$ such that

$$E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h\}\right] \geq E\left[\frac{B^2}{A^2}\right] - \frac{\epsilon'}{3} \tag{C.7}$$

for all $h < h_{\epsilon',2}$. Thus, we can get the lower bound of $Var\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right)$ as

$$\begin{aligned} Var\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right) &= E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h\}\right] - E\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]^2 \\ &\geq E\left[\frac{B^2}{A^2}\right] - \frac{Var\left(\frac{B}{A}\right)}{3} - E\left[\frac{B}{A}\right]^2 - \frac{Var\left(\frac{B}{A}\right)}{3} = \frac{1}{3} Var\left(\frac{B}{A}\right) \end{aligned}$$

where the inequality is due to (C.6) and (C.7) with the choice of $\epsilon' = Var\left(\frac{B}{A}\right) > 0$ under Assumption 1 (iii). Finally, letting $\epsilon = \frac{1}{3} Var\left(\frac{B}{A}\right) > 0$ proves the lemma. □

# D Additional Details for the Discussions of the Assumptions

We first check that Assumption 1 is satisfied in the context of Section 2.5. If $c_0 \neq 0$ then

$$\begin{aligned} \left|E\left[\frac{B}{A}\right]\right| &\geq \left|\int_0^1 \frac{1}{a} c_0 da\right| - \left|\int_0^1 \frac{1}{a} \sum_{\kappa=1}^{\infty} c_\kappa a^\kappa da\right| - \left|\int_1^\infty \frac{1}{a} f_A(a) da\right| \\ &\geq \infty - \sum_{\kappa=1}^{\infty} |c_\kappa| - \int_1^\infty f_A(a) da \geq \infty. \end{aligned} \tag{D.1}$$

If $c_0 = 0$, then

$$\left|E\left[\frac{B}{A}\right]\right| \leq \left|\int_0^1 \frac{1}{a} \sum_{\kappa=1}^{\infty} c_\kappa a^\kappa da\right| + \left|\int_1^\infty \frac{1}{a} f_A(a) da\right| \leq \sum_{\kappa=1}^{\infty} |c_\kappa| + \int_1^\infty f_A(a) da < \infty. \tag{D.2}$$

We next check the statement that $E[B^2/A^2] = \infty$ if and only if $c_1 \neq 0$. If $c_1 \neq 0$, then

$$\begin{aligned} \left|E\left[\frac{B^2}{A^2}\right]\right| &\geq \left|\int_0^1 \frac{1}{a} c_1 da\right| - \left|\int_0^1 \frac{1}{a^2} \sum_{\kappa=2}^{\infty} c_\kappa a^\kappa da\right| - \left|\int_1^\infty \frac{1}{a^2} f_A(a) da\right| \\ &\geq \infty - \sum_{\kappa=1}^{\infty} |c_\kappa| - \int_1^\infty f_A(a) da \geq \infty. \end{aligned} \tag{D.3}$$

If $c_0 = 0$, then

$$\left|E\left[\frac{B^2}{A^2}\right]\right| \le \left|\int_0^1 \frac{1}{a^2}\sum_{\kappa=2}^{\infty} c_\kappa a^\kappa da\right| + \left|\int_1^\infty \frac{1}{a} f_A(a) da\right| \le \sum_{\kappa=2}^{\infty} |c_\kappa| + \int_1^\infty f_A(a) da < \infty. \quad \text{(D.4)}$$

Finally, we check that the variance of the trimmed mean $\widehat{\theta}_h = E_n\left[\frac{B}{A} \cdot \mathbb{1}\{A > h\}\right]$ is $O(\log(1/h))$ as $h = o(1)$ if $c_1 \ne 0$. This follows from

$$\begin{aligned} E\left[\frac{B^2}{A^2} \cdot \mathbb{1}\{A > h\}\right] &= \int_h^1 \frac{1}{a^2}\sum_{\kappa=1}^{\infty} c_\kappa a^\kappa da + \int_1^\infty \frac{1}{a^2} f_A(a) da \\ &= \lim_{k\to\infty} \int_h^1 \frac{1}{a^2}\sum_{\kappa=1}^{k} c_\kappa a^\kappa da + \int_1^\infty \frac{1}{a^2} f_A(a) da \\ &= -c_1 \log(h) + c_2 + \sum_{\kappa=3}^{\infty} c_\kappa (\kappa-1)^{-1}(1 - h^{\kappa-1}) + \int_1^\infty \frac{1}{a^2} f_A(a) da, \end{aligned} \quad \text{(D.5)}$$

where the second equality follows from the dominated convergence theorem with the envelope $a \mapsto \frac{1}{h^2}\sum_{\kappa=1}^{\infty} |c_\kappa|$. Note that $\sum_{\kappa=3}^{\infty} c_\kappa (\kappa-1)^{-1}(1-h^{\kappa-1}) \le \sum_{\kappa=3}^{\infty} |c_\kappa| < \infty$.

# E Proof of the Extended Result: Theorem 3.1

*Proof.* First, Lemma F.2 under Assumptions 1′ (i), 2′ (i)–(ii), and 3′ (i)–(iv) shows that

$$\mu_{h_n}(\gamma_0) = \theta + O(h_n^k) \quad \text{(E.1)}$$

for $h_n \to 0$ as $n \to \infty$. Second, Lemma F.3 under Assumptions 1′ (v), 2′ (iii), and 3′ (iii)–(iv) shows that

$$\widehat{\mu}_{h_n}(\widehat{\gamma}) - \widehat{\mu}_{h_n}(\gamma_0) = \widehat{\mu}_{h_n}^{(1)}(\gamma_0)^T(\widehat{\gamma} - \gamma_0) + o_p(n^{-1/2}) \quad \text{(E.2)}$$

for $nh_n^6 \to \infty$ as $n \to \infty$. Third, Lemma F.4 under Assumptions 2′ (iii) and 3′ (iii), (v) shows that

$$\widehat{\mu}_{h_n}^{(1)}(\gamma_0) = \mu_{h_n}^{(1)}(\gamma_0) + O_p(n^{-1/2}h_n^{-2}) \quad \text{(E.3)}$$

for $nh_n^4 \to \infty$ as $n \to \infty$. Putting (E.1), (E.2), and (E.3) together with Assumption 1 (v), we have

$$\begin{aligned} \widehat{\mu}_{h_n}(\widehat{\gamma}) - \theta &= \widehat{\mu}_{h_n}(\gamma_0) - \mu_{h_n}(\gamma_0) + \mu_{h_n}^{(1)}(\gamma_0)^T(\widehat{\gamma} - \gamma_0) + O(h_n^k) + o_p(n^{-1/2}) \\ &= E_n[Z_n] + O(h_n^k) + o_p(n^{-1/2}) \end{aligned} \quad \text{(E.4)}$$

for $nh_n^6 \to \infty$ as $n \to \infty$.

Next, observe that Assumption 3.1 (ii) implies

$$n^{-1} \cdot Var\left(\frac{Z_n^2}{E[Z_n^2]}\right) \leq n^{-1} \cdot E\left[\frac{Z_n^4}{E[Z_n^2]^2}\right] = n^{-1} \cdot \frac{E[Z_n^4]}{E[Z_n^2]^2} = o(1)$$

as $n \to \infty$. Therefore, Markov's inequality yields

$$\frac{E_n[Z_n^2]}{E[Z_n^2]} - 1 = E_n\left[\frac{Z_n^2}{E[Z_n^2]}\right] - E\left[\frac{Z_n^2}{E[Z_n^2]}\right] = o_p(1) \tag{E.5}$$

as $n \to \infty$. Another implication of Assumption 3.1 (ii) is that it serves as the Lindeberg condition for the triangular array

$$\left\{(n,i) \mapsto n^{-1/2}\frac{Z_{n,i}}{E[Z_n^2]^{1/2}}\middle| i \in \{1,...,n\}, n \in \mathbb{N}\right\},$$

and hence applying Lindeberg-Feller Theorem yields

$$n^{1/2}\frac{E_n[Z_n]}{E[Z_n^2]^{1/2}} \to_d N(0,1) \tag{E.6}$$

as $n \to \infty$.

Finally, observe

$$\begin{aligned} n^{1/2}\widehat{\sigma}_n^{-1}\left(\widehat{\mu}_{h_n}(\widehat{\gamma}) - \theta\right) =& n^{1/2}\frac{E_n\left[Z_n\right] + O(h_n^k) + o_p(n^{-1/2})}{E_n\left[\widehat{Z}_n^2\right]^{1/2}} \\ =& n^{1/2}\frac{E_n\left[Z_n\right] + O(h_n^k) + o_p(n^{-1/2})}{E\left[Z_n^2\right]^{1/2}} \cdot (1 + o_p(1)) \to_d N(0,1) \end{aligned}$$

for $nh_n^6 \to \infty$ and $nh_n^{2k} \to 0$ as $n \to \infty$, where the first equality is due to (E.4), the second equality is due to (E.5) and Lemma F.5 under Assumptions 1′ (ii), (iii), (iv), (v), 2′ (iii) and 3′ (iii), (v), and the last convergence in distribution follows from Assumption 1′ (iii) and (E.6). This completes a proof. □

# F Auxiliary Lemmas for the Extended Result

## F.1 Bias Correction under the Extended Framework

The following lemma provides an extended counterpart of Lemma 2.1.

**Lemma F.1** (Bias Correction)**.** *Suppose that Assumptions 1′ (i), 2′ (i)–(ii), and 3′ (iv) are satisfied. For an integer $k > 1$ provided in Assumption 2′ (ii),*

$$E\left[\widehat{\theta}_h(\gamma_0)\right] - \theta = P_h^{k-1} + O(h^k)$$

*holds as $h \to 0$, where $P_h^{k-1}$ is defined by*

$$P_h^{k-1} = -\sum_{\kappa=1}^{k-1} \frac{h^\kappa}{\kappa!}\left(\frac{1}{\kappa} - s^\kappa\right)\tau_1^{(\kappa)}(0). \tag{F.1}$$

*Proof.* Under Assumptions 1′ (i), 2′ (i)–(ii), and 3′ (iv), we can write

$$\begin{aligned}\theta - E\left[\widehat{\theta}_h(\gamma_0)\right] &= \int_0^h \frac{f_A(a) \cdot E[B|A=a]}{a}\left(1 - S\left(\frac{a}{h}\right)\right) da \\ &= \int_0^h \frac{\tau_1(a) - \tau_1(0)}{a}\left(1 - S\left(\frac{a}{h}\right)\right) da\end{aligned}$$

where the first equality is due to the law of iterated expectations, and the second equality follows from Assumption 2′ (ii). By the $k$-th order mean value expansion under Assumption 2′ (ii), we can write the difference in the last expression as

$$\tau_1(a) - \tau_1(0) = \sum_{\kappa=1}^{k-1} \frac{a^\kappa}{\kappa!}\tau_1^{(\kappa)}(0) + a^k \cdot R^k(\alpha^k(a))$$

with $\alpha^k(a) \in (0, a)$, where the remainder function $R^k$ given by $R^k(a) = \frac{1}{k!}\tau_1^{(k)}(a)$ is uniformly bounded in absolute value on $[0, \eta]$ for some small $\eta > 0$ by Assumption 2′ (ii). Combining the above results together, we can now write

$$\begin{aligned}\theta - E\left[\widehat{\theta}_h(\gamma_0)\right] &= \sum_{\kappa=1}^{k-1} \frac{1}{\kappa!}\tau_1^{(\kappa)}(0)\int_0^h a^{\kappa-1}\left(1 - S\left(\frac{a}{h}\right)\right) da + \int_0^h a^{k-1}R^k\left(\alpha^k(a)\right)\left(1 - S\left(\frac{a}{h}\right)\right) da \\ &= \sum_{\kappa=1}^{k-1} \frac{h^\kappa}{\kappa!}\left(\frac{1}{\kappa} - s^\kappa\right)\tau_1^{(\kappa)}(0) + \int_0^h a^{k-1}R^k\left(\alpha^k(a)\right)\left(1 - S\left(\frac{a}{h}\right)\right) da.\end{aligned}$$

The remaining bias is bounded in absolute value as

$$\begin{aligned}\left|\int_0^h a^{k-1}R^k\left(\alpha^k(a)\right)\left(1 - S\left(\frac{a}{h}\right)\right) da\right| &\leq \int_0^h a^{k-1}\left(1 - S\left(\frac{a}{h}\right)\right) da \sup_{a\in(0,h)}\left|R^k\left(\alpha^k(a)\right)\right| \\ &\leq \frac{1}{k}h^k \sup_{a\in(0,h)}\left|R^k\left(\alpha^k(a)\right)\right|,\end{aligned}$$

where the last inequality is due to Assumption 3′ (iv). Finally, noting that $\sup_{a\in[0,\eta]}\left|R^k(a)\right| < \infty$ proves the claim for $h \leq \eta$. □

## F.2 Bias Estimation under the Extended Framework

The following lemma provides a counterpart of Theorem 2.1.

**Lemma F.2** (Bias Estimation). *If Assumptions 1′ (i), 2′ (i)–(ii), and 3′ (i)–(iv) are satisfied, then*

$$E\left[\widehat{\theta}_h(\gamma_0)\right] - E\left[\widehat{P}_h^{k-1}(\gamma_0)\right] = \theta + O(h^k)$$

*as $h \to 0$.*

*Proof.* First, by the definition of $\widehat{\tau}_1^{(\kappa)}(0)$ given in (3.3), we can write

$$-E[\widehat{P}_h^{k-1}(\gamma_0)] = \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot h^\kappa}{\kappa} E\left[\widehat{\tau}_1^{(\kappa)}(0;\gamma_0)\right] = \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa \cdot h} E\left[B \cdot K^{(\kappa)}\left(\frac{A(\gamma_0)}{h}\right)\right]. \tag{F.2}$$

By Assumptions 2′ (i) and 3′ together with the definition of $\tau_1$ given in (2.4), the last expression in (F.2) may be rewritten as

$$\sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa \cdot h} E\left[B \cdot K^{(\kappa)}\left(\frac{A(\gamma_0)}{h}\right)\right] = \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa \cdot h} \int_0^h \tau_1(a) \cdot K^{(\kappa)}\left(\frac{a}{h}\right) da \tag{F.3}$$

From the proof of Lemma F.1, $\tau_1(a) = \tau_1(a) - \tau_1(0) = \sum_{\kappa=1}^{k-1} \frac{a^\kappa}{\kappa!}\tau_1^{(\kappa)}(0) + a^k \cdot R^k(\alpha^k(a))$ with $\alpha^k(a) \in (0,a)$, where the remainder function $R^k$ given by $R^k(a) = \frac{1}{k!}\tau_1^{(k)}(a)$ is uniformly bounded in absolute value on $[0,\eta]$ for some small $\eta > 0$ by Assumption 2′ (ii). Substituting this mean value expansion in the last expression in (F.3) yields

$$\begin{aligned} &\sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa \cdot h} \int_0^h \tau_1(a) \cdot K^{(\kappa)}\left(\frac{a}{h}\right) da \\ =& \sum_{\kappa_1=1}^{k-1} \frac{h^{\kappa_1}}{\kappa_1!}\tau_1^{(\kappa_1)}(0) \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa} \int_0^1 u^{\kappa_1} K^{(\kappa)}(u)\, du \qquad \text{(F.4)} \\ &+ h^k \sum_{\kappa=1}^{k-1} \frac{\rho_\kappa \cdot (-1)^\kappa}{\kappa} \int_0^1 R^k(\alpha^k(uh)) u^k K^{(\kappa)}(u)\, du, \qquad \text{(F.5)} \end{aligned}$$

where the equality is due to changes of variables. The expression in line (F.4) reduces to

$$\sum_{\kappa_1=1}^{k-1} \frac{h^{\kappa_1}}{\kappa_1!}\left(\frac{1}{\kappa_1} - s^{\kappa_1}\right)\tau_1^{(\kappa_1)}(0) \cdot \left[\left(\frac{1}{\kappa_1} - s^{\kappa_1}\right)^{-1} \sum_{\kappa=1}^{k-1} \rho_\kappa \frac{(-1)^\kappa}{\kappa} \int_0^1 u^{\kappa_1} K^{(\kappa)}(u)\, du\right] = -P_h^{k-1} \tag{F.6}$$

by the definition of $P_h^{k-1}$ and the choice of $\{\rho_\kappa\}_{\kappa=1}^{k-1}$ to satisfy (3.4). To see the asymptotic behavior of line (F.5), note that

$$\begin{aligned}&\left|\sum_{\kappa=1}^{k-1}\frac{\rho_\kappa\cdot(-1)^\kappa}{\kappa}\int_0^1 R^k(\alpha^k(uh))u^k K^{(\kappa)}(u)\,du\right|\\ \leq&\sum_{\kappa=1}^{k-1}\frac{|\rho_\kappa|}{\kappa}\int_0^1 u^k\left|K^{(\kappa)}(u)\right|du\sup_{a\in(0,h)}\left|R^k\left(\alpha^k(a)\right)\right|,\end{aligned}$$

where $\int_0^1 u^k\left|K^{(\kappa)}(u)\right|du<\infty$ for each $\kappa\in\{1,...,k-1\}$ by Assumption 3′, $\sup_{a\in(0,h)}\left|R^k\left(\alpha^k(a)\right)\right|$ is uniformly bounded for $h\in[0,\eta]$, and $0\leq 1-\frac{s^\kappa(h)}{\kappa^{-1}h^\kappa}\leq 1$ under Assumption 3′ (iv). Therefore,

$$h^k\sum_{\kappa=1}^{k-1}\frac{\rho_\kappa\cdot(-1)^\kappa}{\kappa}\int_0^1 R^k(\alpha^k(uh))u^k K^{(\kappa)}(u)\,du=O(h^k)\tag{F.7}$$

as $h\to 0$. Combining the chains of equalities from (F.2)–(F.7), we obtain

$$E[\widehat{P}_h^{k-1}(\gamma_0)]=P_h^{k-1}+O(h^k)\tag{F.8}$$

as $h\to 0$. On the other hand, from Lemma F.1, we also have

$$E\left[\widehat{\theta}_h(\gamma_0)\right]-\theta=\theta_h-\theta=P_h^{k-1}+O(h^k)\tag{F.9}$$

as $h\to 0$. Combining (F.8) and (F.9) yields $E\left[\widehat{\theta}_h(\gamma_0)\right]-E\left[\widehat{P}_h^{k-1}(\gamma_0)\right]=\theta+O(h^k)$ as $h\to 0$. □

## F.3 Auxiliary Lemma: Taylor Expansion

**Lemma F.3.** *If Assumptions 1′ (v), 2′ (iii), and 3′ (iii)–(iv) are satisfied, then*

$$\widehat{\mu}_{h_n}(\widehat{\gamma})-\widehat{\mu}_{h_n}(\gamma_0)=\widehat{\mu}_{h_n}^{(1)}(\gamma_0)^T(\widehat{\gamma}-\gamma_0)+o_p(n^{-1/2})$$

*for $nh_n^6\to\infty$ as $n\to\infty$.*

*Proof.* Assumptions 2′ (iii) and 3′ (iii)–(iv), we can take the second order Taylor expansion

$$\widehat{\mu}_{h_n}(\widehat{\gamma})-\widehat{\mu}_{h_n}(\gamma_0)=\widehat{\mu}_{h_n}^{(1)}(\gamma_0)^T(\widehat{\gamma}-\gamma_0)+\widehat{R}(\gamma_0,\widehat{\gamma}),$$

where, with $\widehat{\mu}_{h_n}^{(2)}$ denoting the Hessian matrix $D^2\widehat{\mu}_{h_n}$, the higher order terms $\widehat{R}(\gamma_0,\widehat{\gamma})$ are bounded as

$$\left|\widehat{R}(\gamma_0,\widehat{\gamma})\right|\leq\sup_{\gamma\in\Gamma}\left|(\widehat{\gamma}-\gamma_0)^T\widehat{\mu}_{h_n}^{(2)}(\gamma)(\widehat{\gamma}-\gamma_0)\right|.$$

To prove the lemma, it suffices to show that right-hand side of the above inequality is $o_p(n^{-1/2})$. To see this, we write

$$\begin{aligned}
&\sup_{\gamma\in\Gamma}\left|(\widehat{\gamma}-\gamma_0)^T\widehat{\mu}^{(2)}_{h_n}(\gamma)(\widehat{\gamma}-\gamma_0)\right| \leq \|\widehat{\gamma}-\gamma_0\|^2\cdot\sup_{\gamma\in\Gamma}\left\|\widehat{\mu}^{(2)}_{h_n}(\gamma)\right\| \\
&\leq \|\widehat{\gamma}-\gamma_0\|^2\cdot E_n\left[|B|\cdot\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)\frac{\omega^{(2)}\left(\frac{A(\gamma)}{h_n}\right)}{h_n^3}A^{(1)}(\gamma)^T+\frac{\omega^{(1)}\left(\frac{A(\gamma)}{h_n}\right)}{h_n^2}A^{(2)}(\gamma)\right\|\right] \\
&\leq \frac{c\cdot\|\widehat{\gamma}-\gamma_0\|^2}{h_n^3}\cdot E_n\left[|B|\cdot\left(\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|+\sup_{\gamma\in\Gamma}\left\|A^{(2)}(\gamma)\right\|\right)\right] \qquad \text{(F.10)}
\end{aligned}$$

for $h_n$ small enough so that $h_n < 1$, where $c = \max\{\left\|\omega^{(1)}\right\|_\infty, \left\|\omega^{(2)}\right\|_\infty\} < \infty$ under Assumption $3'$ (iii), (v). Assumption $2'$ (iii) and applying Khintchin's Weak Law of Large Numbers yield

$$E_n\left[|B|\cdot\left(\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|+\sup_{\gamma\in\Gamma}\left\|A^{(2)}(\gamma)\right\|\right)\right] = O_p(1).$$

Assumption 1 (v) implies $\|\widehat{\gamma}-\gamma_0\|^2 = O_p(n^{-1})$. Therefore, (F.10) can be written as

$$\sup_{\gamma\in\Gamma}\left|(\widehat{\gamma}-\gamma_0)^T\widehat{\mu}^{(2)}_{h_n}(\gamma)(\widehat{\gamma}-\gamma_0)\right| \leq \|\widehat{\gamma}-\gamma_0\|^2\cdot\sup_{\gamma\in\Gamma}\left\|\widehat{\mu}^{(2)}_{h_n}(\gamma)\right\| = O_p(n^{-1}h_n^{-3}),$$

which of $o_p(n^{-1/2})$ for $nh_n^6 \to \infty$ as $n \to \infty$. □

## F.4 Auxiliary Lemma: Consistency of $\widehat{\mu}^{(1)}_{h_n}(\gamma_0)$

**Lemma F.4.** *If Assumptions 2′ (iii) and 3′ (iii), (v) are satisfied, then*

$$\widehat{\mu}^{(1)}_{h_n}(\gamma_0) = \mu^{(1)}_{h_n}(\gamma_0) + O_p(n^{-1/2}h_n^{-2})$$

*for* $nh_n^4 \to \infty$ *as* $n \to \infty$.

*Proof.* We prove the result for each coordinate $\ell$ of $\widehat{\mu}^{(1)}_{h_n}$ and $\mu^{(1)}_{h_n}$. Note that we can write

$$\begin{aligned}
nh_n^4 E\left[\left(\widehat{\mu}^{(1)}_{h_n,\ell}(\gamma_0)-\mu^{(1)}_{h_n,\ell}(\gamma_0)\right)^2\right] &= nh_n^4 Var\left(E_n\left[h_n^{-2}B\omega^{(1)}\left(\frac{A(\gamma_0)}{h}\right)A^{(1)}_\ell(\gamma_0)\right]\right) \\
&= Var\left(B\omega^{(1)}\left(\frac{A(\gamma_0)}{h}\right)A^{(1)}_\ell(\gamma_0)\right) \leq c^2\cdot E\left[B^2A^{(1)}_\ell(\gamma_0)^2\right],
\end{aligned}$$

where $c = \left\|\omega^{(1)}\right\|_\infty < \infty$ under Assumption $3'$ (iii), (v). Note that the last expression is finite under Assumption $2'$ (iii). Finally, applying Markov's inequality yields the desired result. □

## F.5 Auxiliary Lemma: Variance Estimation

**Lemma F.5.** *If Assumptions 1′ (ii), (iii), (iv), (v), 2′ (iii) and 3′ (iii), (v) are satisfied, then*

$$\frac{E_n\left[\widehat{Z}_n^2\right]}{E_n\left[Z_n^2\right]} \to_p 1$$

*for $h_n \to 0$ and $nh_n^6 \to \infty$ as $n \to \infty$.*

*Proof.* For convenience of writing, we introduce the short-hand notation

$$\bar{Z}_n(\gamma) = \left(B \cdot \frac{1}{h_n} \cdot \omega\left(\frac{A(\gamma)}{h_n}\right) - \mu_{h_n}(\gamma)\right) + E_n\left[B \cdot \frac{1}{h_n^2} \cdot \omega^{(1)}\left(\frac{A(\gamma)}{h_n}\right) \cdot A^{(1)}(\gamma)^T\right] \cdot \varphi_0(X).$$

First, we claim $E_n\left[\left(\widehat{Z}_n - \bar{Z}_n(\widehat{\gamma})\right)^2\right] = o_p(1)$. Note that we have

$$\sup_{\gamma\in\Gamma}\left\|\widehat{\mu}_{h_n}^{(1)}(\gamma) - \mu_{h_n}^{(1)}(\gamma)\right\| \leq h_n^{-2}\left(E_n + E\right)\left[\sup_{\gamma\in\Gamma}\left\|B \cdot \omega^{(1)}\left(\frac{A(\gamma)}{h_n}\right) \cdot A^{(1)}(\gamma)\right\|\right] = O_p(h_n^{-2}) \tag{F.11}$$

as $n \to \infty$ by Khintchine's weak law of large numbers under Assumptions 2′ (iii) and 3′ (iii), (v). Since

$$\widehat{Z}_n - \bar{Z}_n(\widehat{\gamma}) = -\left(\widehat{\mu}_{h_n}(\widehat{\gamma}) - \mu_{h_n}(\widehat{\gamma})\right) + E_n\left[B \cdot \frac{1}{h_n^2} \cdot \omega^{(1)}\left(\frac{A(\widehat{\gamma})}{h_n}\right) \cdot A^{(1)}(\widehat{\gamma})^T\right] \cdot \left(\widehat{\varphi}(X) - \varphi(X)\right),$$

applying Minkowski's inequality, Hölder's inequality, and Taylor expansion under Assumptions 2′ (iii) and 3′ (iii), (v) yields

$$\begin{aligned} E_n\left[\left(\widehat{Z}_n - \bar{Z}_n(\widehat{\gamma})\right)^2\right]^{1/2} &\leq \left|\widehat{\mu}_{h_n}(\gamma_0) - \mu_{h_n}(\gamma_0)\right| + \sup_{\gamma\in\Gamma}\left\|\widehat{\mu}_{h_n}^{(1)}(\gamma) - \mu_{h_n}^{(1)}(\gamma)\right\| \cdot \left\|\widehat{\gamma} - \gamma_0\right\| \\ &+ h_n^{-2}\left\|\omega^{(1)}\right\|_\infty \cdot \mathrm{E}_n\left[|B| \cdot \sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)\right\|\right] \cdot E_n\left[\left\|\widehat{\varphi}(X) - \varphi_0(X)\right\|^2\right]^{1/2}. \end{aligned}$$

The first term on the right-hand side is $o_p(1)$ for $nh_n^2 \to \infty$ as $n \to \infty$ by Assumptions 2′ (iii) and 3′ (iii), (v). The second term on the right-hand side is $o_p(1)$ for $nh_n^4 \to \infty$ as $n \to \infty$ by Assumption 1′ (iv) and (F.11). The third term on the right-hand side is $o_p(1)$ for $h_n \to 0$ as $n \to \infty$ by Assumption 1′ (v), 2′ (iii), 3′ (iii), (v). Therefore,

$$E_n\left[\left(\widehat{Z}_n - \bar{Z}_n(\widehat{\gamma})\right)^2\right] = o_p(1) \tag{F.12}$$

for $h_n \to 0$ and $nh_n^4 \to \infty$ as $n \to \infty$.

Second, we claim $E_n\left[\left(\bar{Z}_n(\widehat{\gamma})-\bar{Z}_n(\gamma_0)\right)^2\right]=o_p(1)$. Applying Minkowski's inequality and Hölder's inequality under Assumptions $2'$ (iii) and $3'$ (iii), (v), we obtain

$$\begin{aligned}h_n^3\cdot E_n\left[\sup_{\gamma\in\Gamma}\left\|\bar{Z}_n^{(1)}(\gamma)\right\|^2\right]^{1/2} \leq & h_n\cdot\left\|\omega^{(1)}\right\|_\infty\cdot E_n\left[|B|^2\cdot\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)\right\|^2\right]^{1/2}+h_n^3\cdot\sup_{\gamma\in\Gamma}\left\|\mu_{h_n}^{(1)}(\gamma)\right\| \\ & +\left\|\omega^{(2)}\right\|_\infty\cdot E_n\left[|B|\cdot\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|\right]\cdot \mathrm{E}_n\left[\|\varphi_0(X)\|^2\right]^{1/2} \\ & +h_n\cdot\left\|\omega^{(1)}\right\|_\infty\cdot E_n\left[|B|\cdot\sup_{\gamma\in\Gamma}\left\|A^{(2)}(\gamma)\right\|\right]\cdot \mathrm{E}_n\left[\|\varphi_0(X)\|^2\right]^{1/2}.\end{aligned}$$

The first and second terms on the right-hand side are $O_p(1)$ for $h_n\to\infty$ as $n\to\infty$ by Assumptions $2'$ (iii) and $3'$ (iii), (v). The third and fourth terms on the right-hand side are $O_p(1)$ as $n\to\infty$ by Assumptions $1'$ (v), $2'$ (iii) and $3'$ (iii), (v). Therefore, we have

$$h_n^3\cdot E_n\left[\sup_{\gamma\in\Gamma}\left\|\bar{Z}_n^{(1)}(\gamma)\right\|^2\right]^{1/2}=O_p(1) \tag{F.13}$$

for $h_n\to\infty$ as $n\to\infty$. Applying Taylor expansion, we obtain

$$\begin{aligned}E_n\left[\left(\bar{Z}_n(\widehat{\gamma})-\bar{Z}_n(\gamma_0)\right)^2\right] \leq & E_n\left[\left(\sup_{\gamma\in\Gamma}\bar{Z}_n^{(1)}(\gamma)^T\cdot(\widehat{\gamma}-\gamma_0)\right)^2\right] \\ \leq & E_n\left[\sup_{\gamma\in\Gamma}\left\|\bar{Z}_n^{(1)}(\gamma)\right\|^2\right]\cdot\|\widehat{\gamma}-\gamma_0\|^2=o_p(1)\end{aligned} \tag{F.14}$$

for $h_n\to\infty$ and $nh_n^6\to\infty$ as $n\to\infty$ by (F.13) and Assumption $1'$ (iv).

Third, we claim $E_n\left[\left(Z_n-\bar{Z}_n(\gamma_0)\right)^2\right]=o_p(1)$. Observe that

$$\left\|(E-E_n)\left[B\cdot A^{(1)}(\gamma_0)^T\right]\right\|=O_p(n^{-1/2}) \tag{F.15}$$

as $n\to\infty$ under Assumption $2'$ (iii), and

$$E_n\left[\|\varphi_0(Z)\|^2\right]=O_p(1) \tag{F.16}$$

as $n\to\infty$ under Assumption $1'$ (iv). Note also that

$$\bar{Z}_n(\gamma_0)-Z_n=h_n^{-2}(E_n-E)\left[B\cdot\omega^{(1)}\left(\frac{A(\gamma_0)}{h_n}\right)\cdot A^{(1)}(\gamma_0)^T\right]\cdot\varphi_0(Z).$$

Therefore,

$$E_n\left[\left(Z_n-\bar{Z}_n(\gamma_0)\right)^2\right]\leq h_n^{-4}\cdot\left\|\omega^{(1)}\right\|_\infty^2\cdot\left\|(E-E_n)\left[B\cdot A^{(1)}(\gamma_0)^T\right]\right\|^2\cdot E_n\left[\|\varphi_0(Z)\|^2\right]=o_p(1) \tag{F.17}$$

for $nh_n^4 \to \infty$ as $n \to \infty$ by (F.15), (F.16), and Assumption 3′ (iii), (v).

Fourth, we claim $1/E_n\left[Z_n^2\right] = O_p(1)$. Note that we have $E\left[E_n\left[Z_n^2\right]/E\left[Z_n^2\right]\right] = 1$ and

$$Var\left(\frac{E_n\left[Z_n^2\right]}{E\left[Z_n^2\right]}\right) = \frac{1}{n}Var\left(\frac{Z_n^2}{E\left[Z_n^2\right]}\right) = \frac{1}{n}\frac{E\left[Z_n^4\right]}{E\left[Z_n^2\right]^2} = o(1)$$

by Assumption 1′ (ii). Therefore, by the weak law of large numbers for triangular arrays, we obtain

$$\frac{E_n\left[Z_n^2\right]}{E\left[Z_n^2\right]} = 1 + o_p(1)$$

as $n \to \infty$. From this convergence in probability and Assumption 1′ (iii), we obtain

$$\frac{1}{E_n\left[Z_n^2\right]} = \frac{E\left[Z_n^2\right]}{E_n\left[Z_n^2\right]} \cdot \frac{1}{E\left[Z_n^2\right]} = \frac{1}{1+o_p(1)} \cdot O(1) = O_p(1) \tag{F.18}$$

as $n \to \infty$.

Finally, collecting the above results, we obtain

$$\begin{aligned}\left|\frac{E_n\left[\widehat{Z}_n^2\right]}{E_n\left[Z_n^2\right]} - 1\right| &= \left|\frac{E_n\left[\left(\widehat{Z}_n - Z_n\right)^2\right] + 2E_n\left[Z_n\left(\widehat{Z}_n - Z_n\right)\right]}{E_n\left[Z_n^2\right]}\right| \\ &\leq \frac{E_n\left[\left(\widehat{Z}_n - Z_n\right)^2\right] + 2E_n\left[\left|Z_n\right|\left|\widehat{Z}_n - Z_n\right|\right]}{E_n\left[Z_n^2\right]} \\ &\leq \frac{E_n\left[\left(\widehat{Z}_n - Z_n\right)^2\right]}{E_n\left[Z_n^2\right]} + 2\sqrt{\frac{E_n\left[\left(\widehat{Z}_n - Z_n\right)^2\right]}{E_n\left[Z_n^2\right]}} = o_p(1)\end{aligned}$$

as $n \to \infty$, where the first inequality is due to triangular inequality, the second inequality is due to Cauchy-Schwarz inequality, and the last equality is due to the triangle inequality, (F.12), (F.14), (F.17), and (F.18). □

# G Conditions for Inverse Propensity Score Weighting

In this section, we state a condition under which the implementation procedure (Sections 3.2 and H.3) for the inverse propensity score weighting satisfies Assumptions 1′ and 2′ stated in the general framework (Section 3.1). Concrete examples of the trimming function $S$ and the kernel function $K$ to satisfy Assumption 3′ are proposed in Section H.3.

**Proposition G.1.** *Assumptions 1′ and 2′ hold with $k = 4$ if (i) Assumptions 3′ and 4 hold; (ii) $1/E\left[Z_n^2\right] = O(1)$; (iii) $E[\|W\|^4] < \infty$ and $E[|Y_d|^4] < \infty$ for each $d \in \{0,1\}$; (iv) $E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]$ is finite and invertible; (v) $a \mapsto E[Y_d \mid \pi(W^T\gamma_0) = a^d(1-a)^{1-d}]$ is four times continuously differentiable with bounded derivatives in a neighborhood of 0 for each $d \in \{0,1\}$; and (vi) the density function $f_{W^T\gamma_0}$ for $W^T\gamma_0$ is four times continuously differentiable and satisfies $\lim_{v\to\infty} P(|W^T\gamma_0| \geq v)\exp(4v) < \infty$.*

We remark that the tail condition for $W^T\gamma_0$ stated in part (vi) is satisfied if the distribution function of $W^T\gamma_0$ satisfies $F_{W^T\gamma_0}(v) = 1/(1+\exp(-(v-\mu)/s))$ with $s \leq 1/4$. In particular, the Gaussian distribution satisfies this condition.

*Proof.* We check below that each part of Assumptions 1′ and 2′ is implied by the stated conditions (i)–(vi).

**Assumption 1′ (i):** We have

$$E\left[\left|\frac{B}{A}\right|\right] = E\left[\left|\frac{DY_1}{\pi(W^T\gamma_0)} + \frac{(1-D)Y_0}{\pi(W^T\gamma_0)-1}\right|\right] \leq E\left[\frac{D|Y_1|}{\pi(W^T\gamma_0)} + \frac{(1-D)|Y_0|}{1-\pi(W^T\gamma_0)}\right] = E\left[|Y_1| + |Y_0|\right],$$

where the last equality follows from Assumption 4 (i) stated in condition (i) of the proposition. Therefore, Assumption 1′ (i) is satisfied.

**Assumption 1′ (ii):** Since $nh_n^4 \to \infty$, it suffices to show that $E[(h_nZ_n)^4] = O(1)$ as $n \to \infty$. Note that

$$E[(h_nZ_n)^4]^{1/4} \leq E[|B\omega(A(\gamma_0)/h_n)|^4]^{1/4} + |E[B\omega(A(\gamma_0)/h_n)]| + h_n^{-1} \cdot \|E\left[B\omega^{(1)}(A(\gamma_0)/h_n)A^{(1)}(\gamma_0)\right]\| \cdot E[\|\varphi_0(X)\|^4]^{1/4}.$$

We have $E[|B\omega(A(\gamma_0)/h_n)|^4]^{1/4} + |E[B\omega(A(\gamma_0)/h_n)]| = O(1)$ by conditions (i) and (iii) of the proposition. Since $\varphi_0(X) = E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(D-\pi(W^T\gamma_0))W$, we also have $E[\|\varphi_0(X)\|^4]^{1/4} = O(1)$ by condition (iii) of the proposition. In this light, we will show $\|E\left[B\omega^{(1)}(A(\gamma_0)/h_n)A^{(1)}(\gamma_0)\right]\| = O(h_n)$ as $n \to \infty$. Some calculations yield

$$E\left[B\omega^{(1)}(A(\gamma_0)/h_n)A^{(1)}(\gamma_0)\right] = E\left[DY_1\omega^{(1)}(\pi(W^T\gamma_0)/h_n)\pi^{(1)}(W^T\gamma_0)W\right] - E\left[(1-D)Y_0\omega^{(1)}((1-\pi(W^T\gamma_0))/h_n)\pi^{(1)}(W^T\gamma_0)W\right].$$

We focus on the proof of $E\left[DY_1\omega^{(1)}(\pi(W^T\gamma_0)/h_n)\pi^{(1)}(W^T\gamma_0)W\right] = O(h_n)$ as $n\to\infty$, because the proof for $E\left[(1-D)Y_0\omega^{(1)}((1-\pi(W^T\gamma_0))/h_n)\pi^{(1)}(W^T\gamma_0)W\right]$ is symmetric and similar. Some calculations yield

$$\begin{aligned}
&E\left[DY_1\omega^{(1)}(\pi(W^T\gamma_0)/h_n)\pi^{(1)}(W^T\gamma_0)W\right]\\
=&-h_n^2E\left[(1-\pi(W^T\gamma_0))S(\pi(W^T\gamma_0)/h_n)Y_1W\right]\\
&+h_nE\left[\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))S^{(1)}(\pi(W^T\gamma_0)/h_n)Y_1W\right]\\
&+\sum_{\kappa=1}^{k-1}\frac{\rho_\kappa(-1)^\kappa}{\kappa}E\left[\pi(W^T\gamma_0)^2(1-\pi(W^T\gamma_0))K^{(\kappa+1)}(\pi(W^T\gamma_0)/h_n)Y_1W\right].
\end{aligned}$$

The first two terms on the right-hand side is $O(h_n)$ by conditions (i) and (iii) of the proposition. The last term on the right-hand side is $O(h_n^{3/2})$ because of

$$\begin{aligned}
&E\left[\|\pi(W^T\gamma_0)^2(1-\pi(W^T\gamma_0))K^{(\kappa+1)}(\pi(W^T\gamma_0)/h_n)Y_1W\|\right]\\
\le&E\left[\pi(W^T\gamma_0)^2|K^{(\kappa+1)}(\pi(W^T\gamma_0)/h_n)|\|Y_1W\|\right]\\
\le&\sqrt{E\left[\pi(W^T\gamma_0)^4K^{(\kappa+1)}(\pi(W^T\gamma_0)/h_n)^2\right]E\left[\|Y_1W\|^2\right]},
\end{aligned}$$

condition (iii) of the proposition, and

$$\begin{aligned}
|E\left[\pi(W^T\gamma_0)^4K^{(\kappa+1)}(\pi(W^T\gamma_0)/h_n)^2\right]| &= \int_0^{h_n}p^4K^{(\kappa+1)}(p/h)^2f_{\pi(W^T\gamma_0)}(p)dp\\
&= h_n^3\int_0^1u^4K^{(\kappa+1)}(u)^2f_{\pi(W^T\gamma_0)}(uh)du\\
&\le h_n^3\int_0^1u^4K^{(\kappa+1)}(u)^2du\sup_{p\in(0,1)}|f_{\pi(W^T\gamma_0)}(p)|\\
&= O(h_n^3),
\end{aligned}$$

where $\sup_{p\in(0,1)}|f_{\pi(W^T\gamma_0)}(p)| = \sup_v|f_{\pi(W^T\gamma_0)}(\pi(v))| = \sup_v|f_{W^T\gamma_0}(v)| < \infty$ by condition (vi) of the proposition. Therefore, $E\left[DY_1\omega^{(1)}(\pi(W^T\gamma_0)/h_n)\pi^{(1)}(W^T\gamma_0)W\right] = O(h_n)$ as $n\to\infty$. This completes a proof that Assumption 1′ (ii) is satisfied.

**Assumption 1′ (iii):** Condition (ii) of the proposition implies Assumption 1′ (iii).

**Assumption 1′ (iv):** The maximum likelihood estimator for $\gamma_0$ is defined by

$$\widehat{\gamma} = \max_{\gamma\in\Gamma}E_n[D\log\pi(W^T\gamma)+(1-D)\log(1-\pi(W^T\gamma))]$$

where $\pi(v) = \exp(v)/(1+\exp(v))$. Recall that the influence function for $\gamma$ is $\varphi_0(X) = E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(D-\pi(W^T\gamma_0))W$. Furthermore, its estimator is $\widehat{\varphi}(X) = E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1}(D-\pi(W^T\widehat{\gamma}))W$. The first-order condition reduces to

$$E_n[(D-\pi(W^T\widehat{\gamma}))W] = 0.$$

By Taylor expansion, we can write

$$E_n[(D-\pi(W^T\gamma_0))W] - E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))](\widehat{\gamma}-\gamma_0) + R_n = 0$$

with $\|R_n\| = O_p(\|\widehat{\gamma}-\gamma_0\|^2)$. Given $E_n[(D-\pi(W^T\gamma_0))W] = O_p(n^{-1/2})$ and $\|E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\| = O_p(1)$, we have $\widehat{\gamma}-\gamma_0 = O_p(n^{-1/2})$ and therefore $\|R_n\| = O_p(n^{-1})$. Thus,

$$\begin{aligned}
\widehat{\gamma}-\gamma_0 =& E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(E_n[(D-\pi(W^T\gamma_0))W] + R_n) \\
=& (E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))] + o_p(1))^{-1}(E_n[(D-\pi(W^T\gamma_0))W] + O_p(1/n)) \\
=& E_n[\varphi_0(X)] + o_p(n^{-1/2})
\end{aligned}$$

as $n\to\infty$ under condition (iv) of the proposition. Moreover,

$$\begin{aligned}
E\left[\varphi_0(X)\right] =& E\left[E[E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(D-\pi(W^T\gamma_0))W \mid W]\right] \\
=& E\left[E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(E[D\mid W]-\pi(W^T\gamma_0))W \mid W]\right] = 0
\end{aligned}$$

and

$$\begin{aligned}
E[\|\varphi_0(X)\|^2] =& E[\|E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}(D-\pi(W^T\gamma_0))W\|^2] \\
\leq& E[\|E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\|^2\|W\|^2] \\
=& \|E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\|^2 E[\|W\|^2] < \infty,
\end{aligned}$$

where the last inequality follows from conditions (iii) and (iv) of the proposition.

**Assumption 1′ (v):** Since $\widehat{\varphi}(X)-\varphi_0(X)$ can be written as

$$\begin{aligned}
&(E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1} - E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1})(D-\pi(W^T\gamma_0))W \\
&+ (E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1} - E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1})(D-\pi(W^T\gamma_0))W \\
&- E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1}(\pi(W^T\widehat{\gamma})-\pi(W^T\gamma_0))W,
\end{aligned}$$

we can bound $E_n\left[\|\widehat{\varphi}(X)-\varphi_0(X)\|^2\right]^{1/2}$ by

$$\begin{aligned}
&\|E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1}-E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\|E_n\left[\|W\|^2\right]^{1/2}\\
&+\|E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}-E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\|E_n\left[\|W\|^2\right]^{1/2}\\
&+\|E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1}\|\cdot E_n\left[\|(\pi(W^T\widehat{\gamma})-\pi(W^T\gamma_0))W\|^2\right]^{1/2}.
\end{aligned}$$

The first line of the above term is $O_p(n^{-1/2})$, because

$$\begin{aligned}
&\|E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]-E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]\|\\
&\leq \sup_v|\pi^{(1)}(v)|E_n[\|W\|^3]\|\widehat{\gamma}-\gamma_0\|=O_p(n^{-1/2})
\end{aligned}$$

by condition (iii) of the proposition together with the proof of Assumption $1'$ (iv) above, and

$$\|E_n[WW^T\pi(W^T\widehat{\gamma})(1-\pi(W^T\widehat{\gamma}))]^{-1}-E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\|=O_p(n^{-1/2})$$

by condition (iv) of the proposition. The second term is $O_p(n^{-1/2})$, because

$$\|E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]-E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]\|=O_p(n^{-1/2})$$

under condition (iii) of the proposition, and

$$\|E_n[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}-E[WW^T\pi(W^T\gamma_0)(1-\pi(W^T\gamma_0))]^{-1}\|=O_p(n^{-1/2})$$

by condition (iv) of the proposition. The last term is $O_p(n^{-1/2})$, because

$$\begin{aligned}
\|(\pi(W^T\widehat{\gamma})-\pi(W^T\gamma_0))W\| &\leq \|\pi(W^T\widehat{\gamma})-\pi(W^T\gamma_0)\|\|W\|\\
&\leq \sup_v\|\pi^{(1)}(v)W^T(\widehat{\gamma}-\gamma_0)\|\|W\|\\
&\leq \sup_v\|\pi^{(1)}(v)\|\|\widehat{\gamma}-\gamma_0\|\|W\|^2
\end{aligned}$$

and therefore

$$\begin{aligned}
E_n\left[\|(\pi(W^T\widehat{\gamma})-\pi(W^T\gamma_0))W\|^2\right]^{1/2} &\leq E_n\left[\sup_v\|\pi^{(1)}(v)\|^2\|\widehat{\gamma}-\gamma_0\|^2\|W\|^4\right]^{1/2}\\
&=\sup_v\|\pi^{(1)}(v)\|\|\widehat{\gamma}-\gamma_0\|E_n\left[\|W\|^4\right]^{1/2}\\
&=O_p(n^{-1/2})
\end{aligned}$$

by condition (iii) and the proof of Assumption $1'$ (iv) above.

**Assumption $2'$ (i):** Under condition (vi) of the proposition, the distribution of $A$ is absolutely continuous in a neighborhood of 0, where the density of $A$ is

$$f_A(a) = \frac{f_{W^T\gamma_0}(\pi^{-1}(1-a)) + f_{W^T\gamma_0}(\pi^{-1}(a))}{1-a}.$$

Therefore, Assumption $2'$ (i) is satisfied.

**Assumption $2'$ (ii):** Some calculations yield

$$\tau_1(a) = \frac{E[Y_1 \mid \pi(W^T\gamma_0) = a] f_{W^T\gamma_0}\left(\pi^{-1}(a)\right) - E[Y_0 \mid \pi(W^T\gamma_0) = 1-a] f_{W^T\gamma_0}\left(\pi^{-1}(1-a)\right)}{1-a}.$$

By condition (v) of the proposition, $a \mapsto E[Y_1 \mid \pi(W^T\gamma_0) = a]$ and $a \mapsto E[Y_0 \mid \pi(W^T\gamma_0) = 1-a]$ are four times continuously differentiable with bounded derivatives in a neighborhood of zero. Therefore, it suffices to show that $a \mapsto f_{W^T\gamma_0}\left(\pi^{-1}(a)\right)$ and $a \mapsto f_{W^T\gamma_0}\left(\pi^{-1}(1-a)\right)$ are four times continuously differentiable with bounded derivatives in a neighborhood of zero. We focus on the proof for $f_{W^T\gamma_0}\left(\pi^{-1}(a)\right)$, because the proof for $f_{W^T\gamma_0}\left(\pi^{-1}(1-a)\right)$ is symmetric and similar. Since $\pi^{-1}(a) = \log(a/(1-a))$ and $a \to a/(1-a)$ is four times continuously differentiable near zero, it suffices to show that the mapping $v \mapsto f_{W^T\gamma_0}(\log(v))$ is four times continuously differentiable in a deleted neighborhood of $v = 0$. Calculations yield

$$\begin{aligned}
\frac{\partial}{\partial v} f_{W^T\gamma_0}(\log(v)) &= \frac{f^{(1)}_{W^T\gamma_0}(\log(v))}{v}, \\
\frac{\partial^2}{\partial v^2} f_{W^T\gamma_0}(\log(v)) &= \frac{f^{(2)}_{W^T\gamma_0}(\log(v)) - f^{(1)}_{W^T\gamma_0}(\log(v))}{v^2}, \\
\frac{\partial^3}{\partial v^3} f_{W^T\gamma_0}(\log(v)) &= \frac{f^{(3)}_{W^T\gamma_0}(\log(v)) - 3f^{(2)}_{W^T\gamma_0}(\log(v)) + 2f^{(1)}_{W^T\gamma_0}(\log(v))}{v^3}, \text{ and} \\
\frac{\partial^4}{\partial v^4} f_{W^T\gamma_0}(\log(v)) &= \frac{f^{(4)}_{W^T\gamma_0}(\log(v)) - 6f^{(3)}_{W^T\gamma_0}(\log(v)) + 11f^{(2)}_{W^T\gamma_0}(\log(v)) - 6f^{(1)}_{W^T\gamma_0}(\log(v))}{v^4}.
\end{aligned}$$

These derivatives exist and are continuous in a deleted neighborhood of $v = 0$ by condition (vi) of the proposition. Furthermore, they are bounded near zero under conditions (vi) of the proposition, as

$$\lim_{v\to 0}\frac{f^{(1)}_{W^T\gamma_0}(\log(v))}{v^4}=\lim_{v\to-\infty}\frac{f_{W^T\gamma_0}(v)}{\exp(4v)/4}<\infty,$$
$$\lim_{v\to 0}\frac{f^{(2)}_{W^T\gamma_0}(\log(v))}{v^4}=\lim_{v\to-\infty}\frac{f_{W^T\gamma_0}(v)}{\exp(4v)/16}<\infty,$$
$$\lim_{v\to 0}\frac{f^{(3)}_{W^T\gamma_0}(\log(v))}{v^4}=\lim_{v\to-\infty}\frac{f_{W^T\gamma_0}(v)}{\exp(4v)/64}<\infty,\qquad\text{and}$$
$$\lim_{v\to 0}\frac{f^{(4)}_{W^T\gamma_0}(\log(v))}{v^4}=\lim_{v\to-\infty}\frac{f_{W^T\gamma_0}(v)}{\exp(4v)/256}<\infty.$$

**Assumption 2′ (iii):** Note that $A(\,\cdot\,)$ is twice continuously differentiable with $A^{(1)}(\gamma)=(2D-1)\pi^{(1)}(W^T\gamma)W$ and $A^{(2)}(\gamma)=(2D-1)\pi^{(2)}(W^T\gamma)WW^T$. Furthermore, calculations yield

$$E\left[|B|\cdot\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|\right]\le E\left[|B|\,\|W\|^2\right]\sup_v|\pi^{(1)}(v)|^2,$$
$$E\left[B^2\cdot\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)A^{(1)}(\gamma)^T\right\|\right]\le E\left[B^2\,\|W\|^2\right]\sup_v|\pi^{(1)}(v)|^2,$$
$$E\left[|B|\cdot\sup_{\gamma\in\Gamma}\left\|A^{(1)}(\gamma)\right\|\right]\le E\left[|B|\cdot\|W\|\right]\sup_v|\pi^{(1)}(v)|,\qquad\text{and}$$
$$E\left[|B|\cdot\sup_{\gamma\in\Gamma}\left\|A^{(2)}(\gamma)\right\|\right]\le E\left[|B|\cdot\|W\|^2\right]\sup_v|\pi^{(2)}(v)|.$$

These are bounded under conditions (iii) of the proposition. Therefore, Assumption 2′ (iii) is satisfied. □

# H Guide for Practice

## H.1 Procedural Recipe

We propose a practical recipe for the case of the order $k=3$ and a choice of bandwidth $h_n=O(n^{-1/5})$ based on the mean square error minimization with respect to the second-order bias. These choices are consistent with the rate assumptions stated in the general theory both under regular and irregular cases. Of course, a researcher could make alternative choices of $k$ and bandwidth as far as the rate requirements in the general theory are met.

**Step 1: Bandwidth Choice.** We choose the bandwidth $h_n^*$ by minimizing

$$h^4 \cdot E_n \left[ \frac{B}{2h_{\text{pre}}^3} K^{(2)} \left( \frac{A}{h_{\text{pre}}} \right) \right]^2 + n^{-1} Var_n \left( \frac{B}{A} \cdot \mathbb{1}\{A > h\} - \frac{B}{h \cdot K(0)} \cdot K^{(1)} \left( \frac{A}{h} \right) \right)$$

with respect to $h$, where the preliminary bandwidth $h_{\text{pre}}$ used for a preliminary second-order bias estimation is set to $\max\{A_i\}_{i=1}^n$ for a global estimation.[4]

**Step 2: Bias Corrected Estimation.** Once the bandwidth $h_n^*$ is obtained, the bias-corrected estimate is given by

$$\widehat{\theta}_{h_n^*} - \widehat{P}_{h_n^*}^2 = E_n \left[ \frac{B}{A} \cdot \mathbb{1}\{A > h_n^*\} - \frac{\rho_1 B}{h_n^*} K^{(1)} \left( \frac{A}{h_n^*} \right) + \frac{\rho_2 B}{2h_n^*} K^{(2)} \left( \frac{A}{h_n^*} \right) \right]$$

following (2.2), (2.6), and (2.7), where $\rho_1$ and $\rho_2$ are given following (2.8) by

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -2\int_0^1 uK^{(1)}(u)du & \int_0^1 uK^{(2)}(u)du \\ -2\int_0^1 u^2K^{(1)}(u)du & \int_0^1 u^2K^{(2)}(u)du \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

**Step 3: Variance Estimation.** The variance of $\widehat{\theta}_{h_n^*} - \widehat{P}_{h_n^*}^2$ is approximated by

$$n^{-1} Var_n \left( \frac{B}{A} \cdot \mathbb{1}\{A > h_n^*\} - \frac{\rho_1 B}{h_n^*} K^{(1)} \left( \frac{A}{h_n^*} \right) + \frac{\rho_2 B}{2h_n^*} K^{(2)} \left( \frac{A}{h_n^*} \right) \right)$$

following the result in Section 2.3, where $\rho_1$ and $\rho_2$ are the same as those given in Step 2.

## H.2 Remark on Consistency between Recipe and Theory

The bandwidth choice suggested in Step 1 of Section H.1 induces asymptotically negligible bias relative to the variance when it is used with a bias-corrected inference based on $k = 3$ as in Steps 2 and 3. Specifically, if the identification is regular in the sense that $Var\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\} - \frac{\rho_1 \cdot B}{h} \cdot K^{(1)}\left(\frac{A}{h}\right)\right) \sim 1$ as $h \to 0$, then we have $h_n^* \sim n^{-1/4}$. On the other hand, if the identification is regular in the sense that $Var\left(\frac{B}{A} \cdot \mathbb{1}\{A > h\} - \frac{\rho_1 \cdot B}{h} \cdot K^{(1)}\left(\frac{A}{h}\right)\right) \sim h^{-1}$ as $h \to 0$, then we have the standard nonparametric MSE-optimal rate $h_n^* \sim n^{-1/5}$. In both of these two cases, we can see that the rate requirements $nh_n^2 \to \infty$ and $nh_n^6 \to 0$ as $n \to \infty$ given in Theorem 2.2 for $k = 3$ are satisfied by $h_n = h_n^*$.

[4] This choice of the preliminary bandwidth for the global estimation is analogous to the recommendation by Fan and Gijbels (1996, p. 198) to use a global preliminary parametric estimation for a plug-in optimal bandwidth selection in the context of local polynomial estimation of nonparametric functions.

### H.3 Procedural Recipe for Average Treatment Effect

We propose a practical recipe for inference of average treatment effect following Section 3.2. Our recipe is for the case of order $k = 4$ and a choice of bandwidth $h_n = O(n^{-1/7})$ based on the mean square error minimization with respect to the third-order bias. These choices are consistent with the rate assumptions stated in the theory both under regular and irregular cases.

**Step 1: Logit Parameters.** Estimate $\gamma$ by

$$\widehat{\gamma} = \arg\max_{\gamma\in\Gamma} E_n \left[D \log(\pi(W^T\gamma)) + (1-D)\log(1-\pi(W^T\gamma))\right]$$

where $\pi(W^T\gamma) = \exp(W^T\gamma)/\left(1+\exp(W^T\gamma)\right)$ is the propensity score.

**Step 2: Numerator and Denominator.** Compute

$$A(\widehat{\gamma}) = (2D-1)\cdot \frac{\pi(W^T\widehat{\gamma})\left(1-\pi(W^T\widehat{\gamma})\right)}{D-\pi(W^T\widehat{\gamma})} \qquad \text{and} \qquad B = (2D-1)\cdot Y.$$

**Step 3: Gradient and Influence Function.** Compute

$$\begin{aligned} A^{(1)}(\widehat{\gamma})^T =& (2D-1)\cdot\pi(W^T\widehat{\gamma})\cdot\left(1-\pi(W^T\widehat{\gamma})\right)\cdot W^T \qquad \text{and} \\ \widehat{\varphi}(X) =& E_n\left[W\cdot\pi(W^T\widehat{\gamma})\cdot\left(1-\pi(W^T\widehat{\gamma})\right)\cdot W^T\right]^{-1}\cdot W\cdot(D-\pi(W^T\widehat{\gamma})). \end{aligned}$$

**Step 4: Bandwidth Choice.** Choose the bandwidth $h_n^*$ by minimizing

$$\begin{aligned} & h^6\cdot E_n\left[\frac{B}{3h_{\text{pre}}^4}\cdot K^{(3)}\left(\frac{A(\widehat{\gamma})}{h_{\text{pre}}}\right) + E_n\left[\frac{B}{3h_{\text{pre}}^5}\cdot K^{(4)}\left(\frac{A(\widehat{\gamma})}{h_{\text{pre}}}\right)\cdot A^{(1)}(\widehat{\gamma})^T\right]\cdot\widehat{\varphi}(X)\right]^2 + \\ & n^{-1}Var_n\left(\frac{B}{A(\widehat{\gamma})}\cdot S\left(\frac{A(\widehat{\gamma})}{h}\right) - \frac{\rho_1\cdot B}{h}\cdot K^{(1)}\left(\frac{A(\widehat{\gamma})}{h}\right) + \frac{\rho_2\cdot B}{2h}\cdot K^{(2)}\left(\frac{A(\widehat{\gamma})}{h}\right) + \right. \\ & \qquad E_n\left[\left\{-\frac{B}{A(\widehat{\gamma})^2}\cdot S\left(\frac{A(\widehat{\gamma})}{h}\right) + \frac{B}{A(\widehat{\gamma})\cdot h}\cdot S^{(1)}\left(\frac{A(\widehat{\gamma})}{h}\right)\right.\right. \\ & \qquad\qquad \left.\left.\left. -\frac{\rho_1\cdot B}{h^2}\cdot K^{(2)}\left(\frac{A(\widehat{\gamma})}{h}\right) + \frac{\rho_2\cdot B}{2h^2}\cdot K^{(3)}\left(\frac{A(\widehat{\gamma})}{h}\right)\right\}\cdot A^{(1)}(\widehat{\gamma})^T\right]\cdot\widehat{\varphi}(X)\right), \end{aligned}$$

where the weights are given by

$$\begin{pmatrix}\rho_1 \\ \rho_2\end{pmatrix} = \begin{pmatrix} -\int_0^1 u^1 K^{(1)}(u)du & \frac{1}{2}\int_0^1 u^1 K^{(2)}(u)du \\ -\int_0^1 u^2 K^{(1)}(u)du & \frac{1}{2}\int_0^1 u^2 K^{(2)}(u)du \end{pmatrix}^{-1} \begin{pmatrix} 1-s^1 \\ \frac{1}{2}-s^2 \end{pmatrix}.$$

The preliminary bandwidth $h_{\text{pre}}$ can be chosen to be a benchmark value, such as 0.1 of Crump, Hotz, Imbens and Mitnik (2009). One can repeat this Step 4 by setting $h_{\text{pre}}$ to $h_n^*$

chosen in the first iteration.

**Step 5: Bias Corrected Estimation.** $\widehat{\theta}_{h_n^*}(\widehat{\gamma}) - \widehat{P}^3_{h_n^*}(\widehat{\gamma}) = \widehat{\mu}_{h_n^*}(\widehat{\gamma}) = E_n\left[B \cdot \Omega_{h_n^*}(\widehat{\gamma})\right]$, where

$$\Omega_{h_n^*}(\widehat{\gamma}) = \frac{1}{A(\widehat{\gamma})} \cdot S\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) - \frac{\rho_1}{h_n^*} \cdot K^{(1)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) + \frac{\rho_2}{2h_n^*} \cdot K^{(2)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) - \frac{\rho_3}{3h_n^*} \cdot K^{(3)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right)$$

and the weights are given by

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} -\int_0^1 u^1 K^{(1)}(u)du & \frac{1}{2}\int_0^1 u^1 K^{(2)}(u)du & -\frac{1}{3}\int_0^1 u^1 K^{(3)}(u)du \\ -\int_0^1 u^2 K^{(1)}(u)du & \frac{1}{2}\int_0^1 u^2 K^{(2)}(u)du & -\frac{1}{3}\int_0^1 u^2 K^{(3)}(u)du \\ -\int_0^1 u^3 K^{(1)}(u)du & \frac{1}{2}\int_0^1 u^3 K^{(2)}(u)du & -\frac{1}{3}\int_0^1 u^3 K^{(3)}(u)du \end{pmatrix}^{-1} \begin{pmatrix} 1 - s^1 \\ \frac{1}{2} - s^2 \\ \frac{1}{3} - s^3 \end{pmatrix}.$$

**Step 6: Variance Estimation.** The variance of $\widehat{\theta}_{h_n^*}(\widehat{\gamma}) - \widehat{P}^3_{h_n^*}(\widehat{\gamma}) = \widehat{\mu}_{h_n^*}(\widehat{\gamma})$ is approximated by

$$n^{-1} \cdot E_n\left[\left(B \cdot \Omega_{h_n^*}(\widehat{\gamma}) - \widehat{\mu}_{h_n^*}(\widehat{\gamma}) + E_n\left[B \cdot \Omega^{(1)}_{h_n^*}(\widehat{\gamma}) \cdot A^{(1)}(\widehat{\gamma})^T\right] \cdot \widehat{\varphi}(X)\right)^2\right],$$

where

$$\Omega^{(1)}_{h_n^*}(\widehat{\gamma}) = -\frac{1}{A(\widehat{\gamma})^2} \cdot S\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) + \frac{1}{A(\widehat{\gamma}) \cdot h_n^*} \cdot S^{(1)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) - \frac{\rho_1}{h_n^{*2}} \cdot K^{(2)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) + \frac{\rho_2}{2h_n^{*2}} \cdot K^{(3)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right) - \frac{\rho_3}{3h_n^{*2}} \cdot K^{(4)}\left(\frac{A(\widehat{\gamma})}{h_n^*}\right).$$

and the weights $(\rho_1, \rho_2, \rho_2)^T$ are the same as those given in Step 5.

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# Table

| | | | | RMSE | | | | 95% Coverage | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | dim | $c_\gamma$ | $c_{\beta_d}$ | (I) | (II) | (III) | (IV) | (I) | (II) | (III) | (IV) |
| 500 | 5 | 0.0 | 0.0 | 0.065 | 0.065 | 0.065 | 0.065 | 0.943 | 0.943 | 0.943 | 0.943 |
| 500 | 5 | 1.0 | 0.0 | 0.075 | 0.075 | 0.073 | 0.075 | 0.951 | 0.940 | 0.942 | 0.940 |
| 500 | 5 | 2.0 | 0.0 | 0.119 | 0.113 | 0.107 | 0.143 | 0.944 | 0.925 | 0.836 | 0.930 |
| $n$ | dim | $c_\gamma$ | $c_{\beta_d}$ | (I) | (II) | (III) | (IV) | (I) | (II) | (III) | (IV) |
| 500 | 5 | 0.0 | 0.5 | 0.063 | 0.063 | 0.063 | 0.063 | 0.948 | 0.947 | 0.947 | 0.947 |
| 500 | 5 | 1.0 | 0.5 | 0.078 | 0.077 | 0.074 | 0.078 | 0.950 | 0.940 | 0.940 | 0.941 |
| 500 | 5 | 2.0 | 0.5 | 0.127 | 0.119 | 0.099 | 0.209 | 0.932 | 0.910 | 0.864 | 0.913 |
| $n$ | dim | $c_\gamma$ | $c_{\beta_d}$ | (I) | (II) | (III) | (IV) | (I) | (II) | (III) | (IV) |
| 500 | 10 | 0.0 | 0.5 | 0.064 | 0.064 | 0.064 | 0.064 | 0.951 | 0.948 | 0.948 | 0.948 |
| 500 | 10 | 1.0 | 0.5 | 0.079 | 0.079 | 0.075 | 0.080 | 0.949 | 0.940 | 0.940 | 0.941 |
| 500 | 10 | 2.0 | 0.5 | 0.131 | 0.122 | 0.099 | 0.188 | 0.929 | 0.905 | 0.863 | 0.908 |
| $n$ | dim | $c_\gamma$ | $c_{\beta_d}$ | (I) | (II) | (III) | (IV) | (I) | (II) | (III) | (IV) |
| 1,000 | 10 | 0.0 | 0.5 | 0.045 | 0.045 | 0.045 | 0.045 | 0.942 | 0.942 | 0.942 | 0.942 |
| 1,000 | 10 | 1.0 | 0.5 | 0.055 | 0.055 | 0.053 | 0.055 | 0.947 | 0.942 | 0.943 | 0.944 |
| 1,000 | 10 | 2.0 | 0.5 | 0.092 | 0.085 | 0.082 | 0.135 | 0.937 | 0.908 | 0.793 | 0.917 |

Table 1: Simulation results of the root mean square error (RMSE) and coverage frequency by the estimated 95% confidence interval (95% Coverage) based on 10,000 iterations. Columns indicate (I) our trimmed estimator with optimal bandwidth and bias correction, (II) the trimmed estimator with optimal bandwidth without bias correction, (III) the trimmed estimator with rule-of-thumb bandwidth ($h = 0.1$), and (IV) the untrimmed estimator ($h = 0.0$).