Contests with Non-Specified Success Functions and Nonlinear Spillovers*

Ben Chen† José A. Rodrigues-Neto‡

October 24, 2018

Abstract

Two players in a contest game of complete information simultaneously spend to compete for a prize. They have heterogeneous probability-of-success functions and spillovers. Instead of specifying a functional form for the probability-of-success functions, the proposed game accommodates all functions that satisfy general and reasonable assumptions. These assumptions subsume oft-used functional forms, and capture well-documented behavioral traits that existing models cannot capture. In particular, the game covers the standard case in which the probabilities of success sum to 1, and the non-standard case in which they do not sum to 1. An application of this game captures optimism and pessimism in military conflicts. In equilibrium, the relatively more optimistic contestant makes less military expenses and believes in a greater probability of winning. Another application captures R&D contests with asymmetric firms. In equilibrium, total R&D expenses depend on parameters that measure the spillovers.

Keywords: contest, spillover, behavioral economics, war, R&D.

JEL Classification: C72, D91, H56, O32

*We acknowledge the comments given by Luis Corchón, Larry Samuelson, and James Taylor. All mistakes are our own responsibility.

†Corresponding author. The University of Sydney, Camperdown, NSW - 2006, Australia. URL: `http://nehcneb.com`. E-mail: nehc.neb@gmail.com. This author was partly supported by a Herman N. Finkelstein Memorial Fellowship and an Australian Government Research Training Program Scholarship.

‡Research School of Economics, Australian National University, Canberra, ACT - 2601, Australia. URL: `http://rodriguesneto.weebly.com`. E-mail: jarnwi@yahoo.com.

Contents

1 Introduction

A contest is a game in which each player incurs expenses to compete for a prize. Real-world scenarios answering that description are ubiquitous. A presidential candidate spends on political campaigns to increase her chance of winning the election. A litigant incurs legal expenses to increase her chance of winning the lawsuit. Firms make costly R&D expenses to be the first to obtain a patent and launch an innovative product. Armies employ soldiers and buy weapons to win the war. Contest models have facilitated economic analysis of these and many other scenarios.

A critical component of a contest model is a **success function** that maps the players' strategies into their respective probabilities of winning and losing. The existing contest models typically assume the success function takes a specific functional form (see Serena and Corchón's 2017 survey). While the chosen functional form may appropriately capture some types of contests in the real world, it may poorly capture other contests. It can be hard to ascertain which of the existing functional forms is "ideal". Moreover, while specifying a "nice" functional form can simplify the solution process, the resulting positive predictions and policy recommendations may not be robust to alternative functional forms. To ensure that the implications of the contest model do not depend on the modeler's idiosyncrasies, the theoretical foundations of the contest model should be sufficiently robust.

We therefore propose a general theory of contests. Instead of assuming a particular functional form for the success function, we build a robust theory that allows for a large class of success functions. Imposing only general and reasonable assumptions on the success function *without* specifying its functional form expands the descriptive scope of the model to cover a whole class of contests. Economic analysis premised on a success function that satisfies general assumptions has broader implications than those premised on a success function that takes a specific functional form. The proposed model captures a large class of success functions, including functions frequently used in the existing contest literature, and their convex combinations. Ensuring that the proposed assumptions are closed under finite convex combinations enables the present model to capture uncertainty regarding the success functions themselves. Consider a scenario in which the players do not know which one of a finite collection of success functions will operate to determine their respective probabilities of winning and losing. Suppose the probabilities that each player assigns to these success functions are common knowledge. Subsection 6.1 reveals this scenario falls within the proposed model.

The existing contest models also typically assume that the players have homogeneous success functions (see Serena and Corchón's 2017 survey). While this assumption captures the standard case in which the players believe their probabilities of success sum to 1 and that fact is common knowledge, this assumption does not cover some well-documented behavioral traits. In particular, a body of empirical literature finds that judgments of probabilities commonly exhibit optimism or pessimism (see, for example, Kahneman and Tversky 1977, Radcliffe and Klein 2002). Contest models that assume homogeneous success functions can fail to cover common and important real-world phenomena, and thus give rise to insufficiently robust conclusions.

We introduce and axiomatize a model that allows the players to have heterogeneous success functions. Our formulation covers the standard case in which the probabilities of success functions sum to 1 and that fact is common knowledge. As section 4 will illustrate, the present formulation also captures the non-standard case of players collectively showing optimism (they believe their probabilities of success sum to more than 1) or pessimism (they believe their probabilities of success sum to less than 1). In the non-standard case, each player thinks that she holds correct beliefs and that her opponent is the only optimistic or pessimistic player.

Many real-world contests, such as R&D and litigation, have players whose strategies generate **spillovers** (or externalities) to each other. The existing literature often assumes that spillovers are linear functions of the players' strategies (see, for example, Baye et al. 2012, Chowdhury and Sheremeta 2011a). While assuming linear spillovers has simplification benefits, this assumption can result in poor approximations of spillover functions with high curvatures. Relaxing the linearity assumption, the present model allows spillovers to be homogeneous functions of the players' strategies. In fact, the present model goes further to allow a player to generate spillovers that have *different* degrees of homogeneity in affecting herself and in affecting her opponent (see section 2). The degrees of homogeneity characterize the returns to scale in generating spillovers. Thus, by permitting non-linear spillovers and cross-player differences in returns to scale, the present formulation advances the realism and robustness of contest theory.

In the **Contest Game** set up in section 2, two risk-neutral players simultaneously choose expenses to compete for a prize. Their expenses are inputs in their potentially different success functions, which do not take a functional form. Their expenses also may generate potentially different spillovers to the winner or the loser of the contest. We propose general and reasonable assumptions that roughly require: the players' expenses to be similarly effective in affecting probabilities and in generating spillovers; each player's success function to be increasing with her expenses and sufficiently concave; the spillovers to be sufficiently small so that they alone do not incentivize a player to spend. The players may have different beliefs regarding the success functions, but the parameters and payoff functions are in common knowledge. Section 3 proves the existence of a nontrivial Nash equilibrium comprising positive expenses by both players.

The present assumptions capture well-documented behavioral traits that, to our best knowledge, fall outside the scope of existing contest models. For example, the Conquest Game set up in section 4 specializes the Contest Game to capture optimism and pessimism in military decisionmaking. In the Conquest Game, the winner takes the resources of both players less the dissipation arising from their military expenses and any additional destruction. The players believe that their respective probabilities of success may *not* sum to 1. In the unique equilibrium, the relatively more optimistic player spends relatively *less*, but she believes that she is more likely to win. As the players become collectively more optimistic, their total equilibrium expenses decrease.

The present assumptions subsume oft-used success functions, in particular, the ratio-form Tullock success function that expresses a player's probability of winning as her share of total expenses. Adopting an asymmetric Tullock success function and assuming the players believe their respective probabilities of success sum to 1, the R&D Game set up in section 5 specializes the Contest Game to capture asymmetric advantages in R&D contests. Total equilibrium expenses decrease if the loser receives more spillovers. Moreover, changes in spillovers have a greater impact on total equilibrium expenses if the players' relative advantages become more balanced. This result suggests that legal mechanisms that change R&D spillovers — such as intellectual property law — are more effective in scenarios involving similarly competitive R&D firms than in scenarios involving a dominant firm.

These applications of the Contest Game confirm the benefits of generalizing contest models. Adopting an asymmetric Tullock success function, the R&D Game suggests that the effectiveness of legal mechanism to alter R&D spillovers depends on the balancedness or extremity of the R&D contest. This novel result illustrates that a general contest model, such as the present one, can give rise to novel positive predictions and normative recommendations that the existing, more specialized models cannot obtain (see subsection 5.2). Moreover, the generality of the present model can facilitate verification of whether the positive predictions and normative recommendations obtained in the existing literature remain valid under weaker assumptions. Illustrating this point is a modification of the R&D Game that reveals, for some non-Tullock success function, the conventional wisdom that a more balanced contest leads to greater incentives to spend *does* not hold (see subsection 5.3).

Contest theory traces back to Tullock's (1967, 1980) and Krueger's (1974) analyses of rent-seeking behaviors. Cornes and Hartley (2005), (2012) and others refined Tullock's model to introduce risk aversion and general technologies. Menezes and Quiggin (2010) proved that the standard Tullock contest is strategically equivalent to an oligopsonistic market in which expenditure is the strategic variable. Baye and Hoppe (2003) established that many innovation tournaments and patent-race games are strategically equivalent to rent-seeking contests, including the standard Tullock contest. Building on Katz (1987) and others, Chen and Rodrigues-Neto (2017) constructed a contest model of civil litigation that captures whole classes of success functions and cost-shifting rules that shift a proportion of the winner's legal costs to the loser. Serena and Corchón (2017) offered a recent survey of the contest-theory literature, while Konrad (2009) and Vojnović (2016) provided textbook treatments. Moreover, an extensive experimental literature has developed to document loss aversion and social preferences in contests (recently, Chowdhury et al. 2018), as well as a variety of other non-standard behavioral traits (see Dechenaux et al.'s 2015 survey).

Taking an axiomatic approach to capture both standard and non-standard behavioral traits, this paper builds upon efforts to microfound success functions, especially those originating from Tullock's ratio-form success function.[1] Among the seminal work are Skaperdas's (1996) axiomatization of ratio-form success functions and Clark and Riis's (1998) extension to allow for asymmetric players. Critical to the ratio form is an assumption of independence of irrelevant alternatives (IIA), which, in a two-player model, requires a player to win almost surely if she is the only player who incurs positive expenses. That assumption is not imposed on the present Contest Game (see section 4 and subsection 6.3). The present axiomatization thus offers microfoundations for more flexible success functions, allowing the theory to match a wide range of contests. Indeed, the present axiomatization captures success functions that violate IIA, such as those proposed by Plott (1987) and Beviá and Corchón (2015). However, the present assumptions do not capture difference-form success functions that map differences in the players' strategies into probabilities (for example, Hirshleifer 1989a, Che and Gale 2000).[2] Moreover, the present assumptions do not allow a player's strategy to be a n-tuple vector of non-negative real numbers, while Rai and Sarin's (2009) axiomatization effort does.[3]

Parallel to the Tullock tradition of contest models — in which no player wins a prize almost surely (in equilibrium) no matter how much she spends — are all-pay or rank-order contests (or auctions) — in which the highest spending player wins the highest prize almost surely. Baye et al. (1996) first characterized equilibria for this class of games, and Konrad (2009) and Serena and Corchón (2017) provided recent surveys. Applications of all-pay contests include political lobbying (for example, Hillman and Riley 1989, Baye et al. 1993, Che and Gale 1998), R&D (for example, Che and Gale 2003), litigation (for example, Klemperer 2003, Baye et al. 2005), and school tracking (for example, Xiao 2016). Studying moral-hazard problems faced by firms, Lazear and Rosen (1981) compared workers' incentives under rank-based compensation and under output-based compensation, and Akerlof and Holden (2012) characterized the optimal rank-based compensation structure. To advance their goals, all-pay contest designers can optimally determine the number and distribution of prizes (Moldovanu and Sela 2001), exploit the contestants' concerns

[1]Serena and Corchón (2017) contains a survey of alternative approaches to microfound success functions, such as deriving them as the optimal choice of a contest designer (for example, Corchón and Dahm 2010, Polishchuk and Tonis 2013), exposing their stochastic components (for example, Dixit 1987, Jia 2008), and laying their Bayesian foundations (Skaperdas and Vaidya 2012).

[2]Corchón's (2007) survey revealed the extent to which difference-form success functions admit a Nash equilibrium in common scenarios.

[3]A contest model in which a player's strategy is a vector of two variables can capture her spending of resources to increase her probability of success and to sabotage her opponent. For a survey of sabotage in contests, see Chowdhury and Gürtler (2015).

for relative ranking (Moldovanu et al. 2007), or introduce insurance to reimburse the losers (Minchuk and Sela 2017). While the present effort clearly does not attempt to develop all-pay contests, we nonetheless share the same ambition as authors who generalize all-pay contests. In particular, Baye et al. (2012) offer equilibrium characterization of a simultaneous-move, two-player rank-order contests with complete information, in which each player generates affine spillovers that depend on her rank. Siegel (2009, 2010) generalizes all-pay contests to allow for arbitrary cost functions, while Xiao's (2016) model permits convex prize sequences. Moreover, Olszewski and Siegel (2016) approximate the equilibrium outcomes of all-pay contests with a large number of asymmetric players who may have complete or incomplete information. Xiao (2017) recently builds upon Siegel (2009, 2010) to introduce performance spillovers in the sense of a player's strategy entering into the other players' cost functions. Sacco and Schmutzler (2008), Bos and Ranger (2014) and Chowdhury (2017) also model all-pay auctions with spillovers in the sense of the winner's bid size affecting her prize of winning.

Section 2 constructs the Contest Game. Section 3 finds and characterizes an equilibrium with positive expenses by both players. Section 4 applies the Contest Game to model optimism and pessimism in military decisionmaking, assuming the players believe their respective probabilities of success may *not* sum to 1. Section 5 applies the Contest Game to model asymmetric advantages in R&D contests, in which the probabilities of success sum to 1 but the players are asymmetrically productive. Section 6 reveals the extent to which the Contest Game captures: uncertainty regarding the success functions; homogeneous expenses and spillovers; and Tullock contest models. Section 7 explores future research directions. Appendix A contains all proofs.

2 The Contest Game

The **Contest Game** is a simultaneous-move game of complete information characterized by two risk-neutral players 1 and 2, their common action space $\mathbb{R}_+$, and their payoff functions U_1, U_2. The payoff functions and parameters are common knowledge.

Let $i \in \{1,2\}$ represent a generic player and $j \in \{1,2\} \setminus \{i\}$ her opponent. Players i, j simultaneously choose $e_i, e_j \geq 0$ levels of expenses. Player i values the prize at $v_i > 0$. A twice-continuously-differentiable **success function** $\theta_i : \mathbb{R}_+^2 \longrightarrow [0,1]$ that satisfies additional assumptions to be set out below gives $\theta_i(e_i, e_j)$ as player i's probability of winning a prize according to her own belief; she believes that her opponent j wins the prize with complementary probability $1 - \theta_i$. Thus player i believes her and the opponent j's respective probabilities of winning sum to 1.

Remark 1. *It is possible to have $\theta_1 + \theta_2 \neq 1$. We interpret the scenario of $\theta_1 + \theta_2 \neq 1$ as capturing non-common beliefs regarding a "true" success function. Suppose there is a collection of success* *functions containing* θ_1, θ_2. *Before the players choose their expenses, Nature chooses one of these success functions to determine the outcome of the contest. At the time of choosing their respective expenses, both players know the collection of success functions, but they do not observe Nature's choice. Player* 1 *assigns probability* 1 *to the event of Nature choosing* θ_1, *while player* 2 *assigns probability* 1 *to Nature choosing* θ_2. *See subsection 6.1 for an alternative scenario.*

Both the winner and the loser may experience spillovers. Exogenous scalars $w_{ii}, l_{ii} \in \mathbb{R}$ respectively characterize the **winner's spillovers** and **loser's spillovers** that player i's expenses e_i generates to herself. Player i receives (self-generated) winner's spillovers $w_{ii}e_i$ if she wins, and loser's spillovers $l_{ii}e_i$ if she loses.

Player i also may receive winner's and loser's spillovers generated by her opponent j's expenses e_j. Exogenous scalars $w_{ij}, l_{ij} \in \mathbb{R}$ and $k_i > 0$, $i \neq j$, characterize these spillovers. Player i receives winner's spillovers $w_{ij}e_j^{k_i}$ if she wins, and loser's spillovers $l_{ij}e_j^{k_i}$ if she loses. The exponent k_i characterizes the returns of e_j in generating spillovers to player i.[4] Example 1 contains a case where $k_i \neq 1$. Call $e_j^{k_i}$ **player j's effective expenses affecting player i**, and e_i **player i's effective expenses affecting herself**. Thus player i receives self-generated, intra-player spillovers ($w_{ii}e_i$ and $l_{ii}e_i$) and opponent-generated, inter-player spillovers ($w_{ij}e_j^{k_i}$ and $l_{ij}e_j^{k_i}$).[5]

Player i's **payoff** is $U_i : \mathbb{R}_+^2 \to \mathbb{R}$ given by

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)\left[v_i + w_{ii}e_i + w_{ij}e_j^{k_i}\right] + \left[1 - \theta_i(e_i, e_j)\right]\left[l_{ii}e_i + l_{ij}e_j^{k_i}\right] - e_i, \tag{1}$$

which is her expected outcome measured in monetary terms. She believes that if she wins, then she receives the sum of her prize value and her winner's spillovers, $v_i + w_{ii}e_i + w_{ij}e_j^{k_i}$. She believes that if she loses, then she receives the sum of her loser's spillovers, $l_{ii}e_i + l_{ij}e_j^{k_i}$. Weights $\theta_i(e_i, e_j)$, $1 - \theta_i(e_i, e_j)$ are respectively her probabilities of winning and losing according to her belief. Because she incurs expenses e_i whether she wins or loses, the probability weights do not scale e_i. This specification for the payoff function assumes additive separability of expenses, spillovers and prizes. This means that expenses, spillovers and prizes are perfect substitutes and the players are risk neutral.

We now state and impose the following Assumptions 1-6 to guarantee equilibrium existence.

Assumption 1. *Player i's success function satisfies* $\theta_i(e_i, e_j) = \theta_i(x^{k_i}e_i, xe_j)$ *for any scalar* $x > 0$.

Assumption 1 requires player i to believe that the same ratio of effective expenses — $e_i/e_j^{k_i}$ and $x^{k_i}e_i/(xe_j)^{k_i}$ for $x > 0$ — leads to the same probabilities of winning.[6] Intuitively, Assumption 1

[4]As subsection 6.2 will reveal, the present specification that each player's expenses linearly produce spillovers to herself captures the more general case of these spillovers being homogeneous functions.

[5]Sections 4 and 5 will utilize the distinction between these two notions of spillovers to capture real-world scenarios.

[6]Section 3 will offer diagrammatic representations of Assumption 1.

requires e_i and e_j to be similarly effective in affecting probabilities of winning and in producing spillovers. Because scalars 1 and k_i respectively characterize the returns of e_i and e_j in generating spillovers to player i, Assumption 1 ensures that 1 and k_i also respectively characterize the returns of e_i and e_j in affecting her belief regarding her probability of winning; player i's belief regarding how expenses affect her spillovers and probability of winning is thus consistent in this sense. Moreover, we aim to define a class of success functions that includes Tullock success functions, because they are the standard in contest theory (see section 1 and subsection 6.3). In the special case of $k_i = 1$, Assumption 1 reduces to the homogeneity of degree zero property, which is satisfied by Tullock success functions.

Assumption 2. *Player i's success function θ_i satisfies the following properties:*

1. *Her marginal probability of success with respect to her own expenses e_i is positive and non-increasing, and is bounded above. Formally, for a positive upper bound $\bar{b}$ which may depend on her opponent's expenses e_j,*

$$0 < \frac{\partial \theta_i}{\partial e_i} \leq \bar{b}, \qquad\qquad \frac{\partial^2 \theta_i}{\partial e_i^2} \leq 0.$$

2. *Her marginal probability of success with respect to her opponent's effective expenses $e_j^{k_i}$ is non-decreasing, and is bounded below. Formally, for a lower bound $\underline{b}$ which may depend on e_i,*

$$\frac{\partial \theta_i}{\partial e_j^{k_i}} \geq \underline{b}, \qquad\qquad \frac{\partial^2 \theta_i}{\left(\partial e_j^{k_i}\right)^2} \geq 0.$$

Together with Assumption 5 below, the upper and lower bounds that Assumption 2 impose prevent player i from have incentives to make unbounded expenses.

Define **spillover differentials** $\delta_{ii}, \delta_{ij} \in \mathbb{R}$ by $\delta_{ii} = w_{ii} - l_{ii}$ and $\delta_{ij} = w_{ij} - l_{ij}$ respectively. The value $\delta_{ii} e_i$ (respectively, $\delta_{ij} e_j^{k_i}$) is the difference between the winner's and loser's spillovers of player i arising from her own expenses e_i (her opponent's expenses e_j). Rearrange player i's payoff U_i defined by (1) to obtain

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)\Big[v_i + \delta_{ii} e_i + \delta_{ij} e_j^{k_i}\Big] - (1 - l_{ii}) e_i + l_{ij} e_j^{k_i}. \tag{2}$$

Equation (2) expresses player i's payoff as her expected benefits of winning — the weighted sum of her prize value and spillover differentials, $\theta_i(e_i, e_j)\Big[v_i + \delta_{ii} e_i + \delta_{ij} e_j^{k_i}\Big]$ — less her *unweighted* expenses $(1 - l_{ii}) e_i$, and plus a term $l_{ij} e_j^{k_i}$ that does not vary with her own expenses e_i. The loser's spillovers $l_{ii}e_i$ reduce her unweighted expenses because $\delta_{ii}e_i$ gives the *additional* spillovers of winning that her own expenses produce.

Assumption 3. *Player i believes her spillovers are sufficiently small, in the following sense:*

$$\delta_{ii}\theta_i(e_i, e_j) + \left[\delta_{ii}e_i + \delta_{ij}e_j^{k_i}\right]\frac{\partial\theta_i}{\partial e_i} < 1 - l_{ii}, \tag{3}$$

where the inequality holds strictly in the limit when $e_i \to 0+$.

Assumption 3 ensures that player i would have no incentives to make positive expenses if the (non-spillover) prize of winning had zero value to her ($v_i = 0$). Inequality (3) comes from taking the partial derivative of her payoff U_i in equation (2) with respect to her own expenses e_i, and assuming $v_i = 0$. The left-hand side of inequality (3) sums her marginal (expected) benefits of making positive expenses, while the right-hand side her marginal unweighted expenses. Inequality (3) ensures her marginal unweighted expenses to be greater than her marginal benefits if $v_i = 0$.

Assumption 4. *Player i's success function θ_i is sufficiently concave in the following sense:*

$$\frac{\frac{\partial^2}{\partial e_i^2}\left[\frac{\theta_i(e_i,e_j)}{1-l_{ii}-\delta_{ii}\theta_i(e_i,e_j)}\right]}{\frac{\partial}{\partial e_i}\left[\frac{\theta_i(e_i,e_j)}{1-l_{ii}-\delta_{ii}\theta_i(e_i,e_j)}\right]} < 0.$$

Assumption 4 is a technical assumption that ensures player i's payoff is strictly quasiconcave in her own expenses.

Assumption 5. *Player i believes that her success function and self-generated spillovers satisfy the following properties:*

1. *Her self-generated spillovers do not exceed her expenses. Formally,* $1 \geq \max\{w_{ii}, l_{ii}\}$.
2. *In the special case of her self-generated winner's (respectively, loser's) spillovers covering her expenses, she does not win (lose) almost surely by making infinitely more (less) effective expenses than her opponent j does. Formally,*

$$1 = w_{ii} \quad \Rightarrow \quad \lim_{e_i/e_j^{k_i} \to +\infty} \theta_i(e_i, e_j) < 1,$$

$$1 = l_{ii} \quad \Rightarrow \quad \lim_{e_i/e_j^{k_i} \to 0+} \theta_i(e_i, e_j) > 0.$$

Assumption 5 prevents player i from recouping all expenses by generating spillovers alone. Part 1 bounds her marginal benefit of generating spillovers above by 1, her marginal cost of spending.

In the special case where her spillovers cover her expenses if she wins (respectively, loses), part 2 prevents her from recovering all expenses almost surely by making infinitely more (less) effective expenses than her opponent does. Assumption 5 is trivially satisfied if her expenses strictly exceed her spillovers, $1 > \max\{w_{ii}, l_{ii}\}$.

Assumption 6. *The players' expenses are insufficiently effective in affecting each other, in the sense of* $k_1 k_2 \leq 2$.

Assumption 6 restricts the effectiveness of the players' expenses in affecting each other. Assumption 6 is trivially satisfied in the special case where each player's expenses linearly affect her opponent, $k_1 = k_2 = 1$. In general, the product of k_1 and k_2 must be sufficiently small (in the sense of $k_1 k_2 \leq 2$) to satisfy Assumption 6. Intuitively, Assumption 6 facilitates equilibrium existence by ruling out significant divergence in relative effective expenses. In all that follows, the previous assumptions hold.

The solution concept adopted is a Nash equilibrium that is **nontrivial** in the sense of comprising positive expenses by both players.

3 Equilibrium

This section proves the existence of a nontrivial Nash equilibrium, and provides a characterization of it.

Lemmas 1 and 2 together allow each player's best response to be characterized by her FOC. Appendix A contains all proofs.

Lemma 1. *Player i's payoff function* U_i *is strictly quasiconcave in her own expenses,* e_i.

Lemma 2. *Suppose some* $e_i' > 0$ *satisfies the FOC for player i's payoff function* U_i *restricted to a function of one variable, her expenses* e_i. *Then* e_i' *is a global maximum for the same restriction of* U_i.

Lemma 1 implies a player's locally-optimal strategy is globally-optimal, holding her opponent's strategy fixed. Lemma 2 then allows a player's FOC to characterize her optimal strategy. Thus Lemmas 1 and 2 together imply that a pair of positive strategies constitutes a nontrivial Nash equilibrium if and only if it satisfies the following system of FOCs:

$$\begin{cases} \frac{\partial U_1}{\partial e_1} = \frac{\partial \theta_1}{\partial e_1}\left[v_1 + \delta_{11} e_1 + \delta_{12} e_2^{k_1}\right] + \delta_{11}\theta_1(e_1, e_2) - (1 - l_{11}) = 0 \\ \frac{\partial U_2}{\partial e_2} = \frac{\partial \theta_2}{\partial e_2}\left[v_2 + \delta_{21} e_1^{k_2} + \delta_{22} e_2\right] + \delta_{22}\theta_2(e_2, e_1) - (1 - l_{22}) = 0. \end{cases} \tag{4}$$

where the first two terms in each player's FOC are her marginal benefits (weighted by her probability of winning), and the last term her marginal expenses (unweighted).

We now use Assumption 1 to obtain and characterize the relative effective expenses in any potential equilibrium. Assumption 1 implies that, given a positive constant r_{ii}, player i believes she has the same probability of success $\theta_i(e_i, e_j)$ under any pair of positive expenses (e_i, e_j) satisfying $e_i/e_j^{k_i} = r_{ii}$; some algebra using Assumption 1 reveals

$$\theta_i(e_i, e_j) = \theta_i(r_{ii}, 1) = \theta_i\left(1, r_{ii}^{-1/k_i}\right). \tag{5}$$

If $e_i, e_j > 0$, then $r_{ii} = e_i/e_j^{k_i}$ captures player i's effective expenses *ratio* affecting her, while $r_{ij} = 1/r_{ii}$ captures her opponent j's effective expenses ratio affecting player i. A pair of positive constants (r_{ii}, k_i) thus characterizes a class of positive expenses pairs inducing the same probability of success according to player i's belief. Slightly abusing notation, we utitlize equation (5) to denote

$$\theta_i(r_{ii}) = \theta_i(r_{ii}, 1), \qquad\qquad \theta_i^{(i)}(r_{ii}) = \frac{\partial}{\partial r_{ii}}\theta_i(r_{ii}).$$

Example 1. *Illustrating Assumption 1 is the following modification of the Tullock success function:*

$$\theta_1(e_1, e_2) = \begin{cases} \frac{e_1}{e_1 + e_2^{k_1}} & \text{if } e_1 + e_2 > 0, \\ \frac{1}{2} & \text{otherwise,} \end{cases} \tag{6}$$

where k_1 also captures the effectiveness of player 2's expenses in producing spillovers to player 1 (see player 1's payoff as defined by equation (1)).

Figure 1 depicts three classes of positive expenses pairs, where each class induces the same effective expenses ratio and probabilities according to player 1's success function defined by (6). The black solid curve captures the class inducing $r_{11} = 1$ and $\theta_1 = \frac{1}{2}$ when $k_1 = 2$; this is the case of player 1 believing that, given the returns to her expenses are half of player 2's, her effective relative expenses ratio is 1 and her probability of success $\frac{1}{2}$. The blue dotted curve represents the class of positive expenses pairs inducing $r_{11} = 1$ and $\theta_1 = \frac{1}{2}$ when $k_1 = 1$, while the red dashed curve the class inducing $r_{11} = 3$ and $\theta_1 = \frac{3}{4}$ when $k_1 = 2$.

Lemma 3 will find a pair of positive effective expenses ratios which will be used to characterize a nontrival Nash equilibrium. For each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{1\}$, define a function ϕ_{ij} :

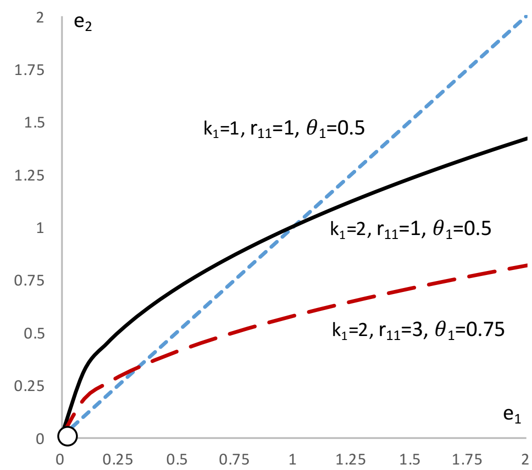

Figure 1: Positive expenses pairs that induce the same probabilities according to success function (6).

$\mathbb{R}_{++} \to \mathbb{R}_{++}$ by[7]

$$\phi_{ij}(r_{ii}) = \frac{v_i \theta_i^{(i)}(r_{ii})}{1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}) - \left(\delta_{ii} r_{ii} + \delta_{ij}\right)\theta_i^{(i)}(r_{ii})}. \tag{7}$$

Lemma 3. *There exists a pair of positive constants* $\left(r_{11}^*, r_{22}^*\right)$ *that satisfies the following properties:*

1. *If* $k_1 k_2 = 1$, *then*

$$r_{11}^* r_{22}^{*k_1} = 1, \qquad \left[r_{11}^* \phi_{12}(r_{11}^*)\right]^{k_2} = \phi_{21}(r_{22}^*).$$

2. *If* $k_1 k_2 \neq 1$, *then*

$$r_{11}^{*k_1 k_2} r_{22}^{*k_1} = \left[\phi_{12}(r_{11}^*)\right]^{1-k_1 k_2}, \qquad r_{22}^{*k_1 k_2} r_{11}^{*k_2} = \left[\phi_{21}(r_{22}^*)\right]^{1-k_1 k_2}.$$

Proposition 1 will reveal that the constant r_{11}^* (respectively, r_{22}^*) that Lemma 3 characterizes is player 1's (player 2's) relative expenses ratio affecting her in a nontrivial Nash equilibrium.[8]

Proposition 1. *The Contest Game has a nontrival Nash equilibrium* $\left(e_1^*, e_2^*\right) \in \mathbb{R}_{++}^2$ *characterized by*

$$e_1^* = r_{11}^* \phi_{12}(r_{11}^*) \qquad e_2^* = r_{22}^* \phi_{21}(r_{22}^*)$$

[7]Lemma 4 in Appendix A proves that function ϕ_{ij} is strictly positive.

[8]The proof of Proposition 1 reveals that $\phi_{12}(r_{11}^*)$ (respectively, $\phi_{21}(r_{22}^*)$) is player 2's (player 1's) relative expenses ratio affecting player 1 (player 2) in equilibrium.

where Lemma 3 defines $\left(r_{11}^{*}, r_{22}^{*}\right) \in \mathbb{R}_{++}^{2}$.

Proposition 1 finds and characterizes a nontrivial Nash equilibrium $\left(e_1^{*}, e_2^{*}\right)$, which is a function of the pair of positive expenses ratios $\left(r_{11}^{*}, r_{22}^{*}\right)$ that Lemma 3 characterizes. Upon specifying the functional forms of the success functions, an application of Lemma 3 and Proposition 1 will give close-form expressions for the equilibrium expenses; sections 4 and 5 will specify the functional forms of the success functions to study military conflicts and R&D.

Corollary 1. *If Lemma 3 defines a unique pair of positive constants, then Proposition 1 characterizes the unique nontrival Nash equilibrium of the Contest Game.*

Corollary 1 proves that a sufficient condition for the uniqueness of a nontrival Nash equilibrium is Lemma 3 finding a unique pair $\left(r_{11}^{*}, r_{22}^{*}\right)$. Hence adding assumptions to restrict the success functions and spillover functions to ensure a unique $\left(r_{11}^{*}, r_{22}^{*}\right)$ will guarantee equilibrium uniqueness.[9]

Example 2. *To illustrate closed-form expressions for an equilibrium of the Contest Game, consider a special case with the following features:*

1. *Player* 1 *values the (non-spillover) prize at* $v_1 = 1$, *while player* 2 *values it at* v_2.[10]
2. *Both players' expenses* e_1, e_2 *generate no spillovers,* $w_{ii} = l_{ii} = w_{ij} = l_{ij} = 0$.
3. *If* $e_1 + e_2 > 0$, *then players* 1 *and* 2*'s respective success functions* θ_1, θ_2 *have the following functional forms*

$$\theta_1(e_1, e_2) = \frac{e_1}{e_1 + e_2}, \qquad \theta_2(e_2, e_1) = \frac{e_2}{e_1^2 + e_2}.$$

   If $e_1 + e_2 = 0$, *then* $\theta_1(e_1, e_2) = \theta_2(e_2, e_1) = \frac{1}{2}$.

4. *Players* 1 *and* 2*'s payoff functions* U_1, U_2 *are respectively*

$$U_1(e_1, e_2) = \theta_1(e_1, e_2) - e_1, \qquad U_2(e_2, e_1) = \theta_2(e_2, e_1)v_2 - e_2.$$

To find a nontrivial Nash equilibrium directly would require solving the following system (8) of equations that has a three-degree polynomial:

$$\begin{cases} e_1^3 - 2(1 + \sqrt{v_2})e_1^2 + \left(2 + v_2 + 2\sqrt{v_2}\right)e_1 - \sqrt{v_2} = 0 \\ e_2 = \left(\sqrt{v_2} - e_1\right)e_1. \end{cases} \tag{8}$$

[9]For an example of a generally-formulated contest model of civil litigation that has a unique nontrivial Nash equilibrium, see Chen and Rodrigues-Neto (2017). For conditions under which multiple equilibrium exist in Tullock contests, see Chowdhury and Sheremeta (2011b).

[10]Assumption $v_1 = 1$ is not a mere normalization assumption. This is due to the non-linearity of beliefs.

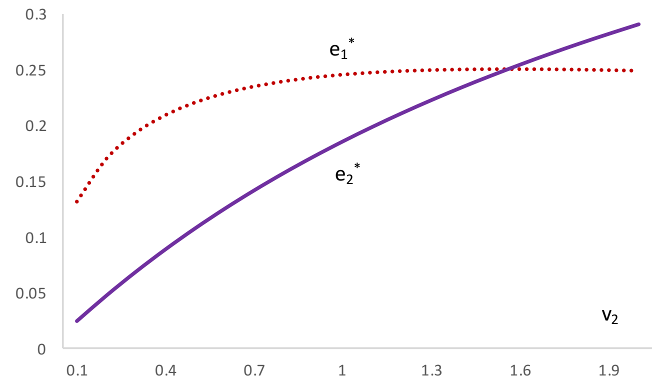

Figure 2: Equilibrium expenses in Example 2.

Alternatively, applying Lemma 3 and Proposition 1 finds a nontrivial Nash equilibrium indirectly. To see this, first apply Lemma 3 to find a unique pair of positive constants (r_{11}^*, r_{22}^*) *satisfying*[11]

$$\begin{cases} \sqrt{v_2} r_{11}^{*3} + 2(\sqrt{v_2} - 1) r_{11}^{*2} + (\sqrt{v_2} - 2) r_{11}^* - 1 = 0 \\ r_{22}^* = \left(1 + r_{11}^{*-1}\right)^2. \end{cases} \tag{9}$$

An application of Proposition 1 then finds the corresponding nontrivial Nash equilibrium:

$$e_1^* = \frac{r_{11}^*}{\left(1 + r_{11}^*\right)^2}, \qquad e_2^* = v_2 \left(\frac{r_{11}^* \left(1 + r_{11}^*\right)}{r_{11}^{*2} + \left(1 + r_{11}^*\right)^2} \right)^2.$$

Figure 2 depicts the equilibrium expenses under different values of v_2*. The red dotted and purple solid curves respectively indicate players* 1 *and* 2*'s equilibrium expenses. For instance, if* $v_2 = 25/16$*, then* $r_{11}^* = 1$*,* $r_{22}^* = 4$*, and* $e_1^* = e_2^* = 1/4$*.*

4 Application: Optimism and Pessimism in Military Conflicts

This section applies the Contest Game to study optimism and pessimism in military conflicts, in particular, the scenario of two warring states spending resources in order to conquer each other. In this scenario, each state makes costly military expenses with its own resources, and the winner keeps its own remaining resources and takes over the loser's remaining resources. Dissipating the resources that the winner obtains, the loser's expenses create negative (cross-player) spillovers.

[11] The coefficients in the three-degree polynomial in system (9) change sign exactly once, because it is impossible to have $\sqrt{v_2} - 1 < 0$ and $\sqrt{v_2} - 2 > 0$. Then Descarte's sign rule implies that this polynomial has a unique real root.

Hirshleifer (1989b) studied these military conflicts with Tullock contest models, and Garfinkel and Skaperdas (2000) used a two-period Tullock model to reveal that a player may make military expenses in order to weaken her opponent in the future. Introducing private information regarding the valuation of the resources at stake, Corchón and Yıldızparlak (2013) considered the equilibrium properties of a two-period Tullock model in which the declaration of war in the first period signals information. This subsection introduces a (one-period) contest model that captures optimism and pessimism, which are well-documented behavioral traits (see, for example, Kahneman and Tversky 1977; Weinstein 1980; Radcliffe and Klein 2002; Puri and Robinson 2007). Military texts and history have long recognized the importance of accounting for optimism and pessimism in military decisionmaking.[12]

4.1 Setup

Construct the **Conquest Game** by specializing the following features of the Contest Game:

1. The players value the (non-spillover) prize of winning equally; $v_1 = v_2 = v$ for a constant $v > 0$. The prize is interpreted as the sum of the players' individual pre-war resources, $0.5v$. The winner will obtain v by conquest.
2. Player i's success function θ_i has the following functional form:[13]

$$\theta_i(e_i, e_j) = \begin{cases} \eta\mu_i + (1-\eta)\frac{e_i}{e_1+e_2} & \text{if } e_1 + e_2 > 0, \\ \mu_i & \text{otherwise}, \end{cases} \tag{10}$$

   where constants $\mu_i, \eta \in (0, 1)$ respectively capture player i's belief regarding her relative advantages in winning the war, and the weight she assigns to μ_i rather than her share of total expenses, $e_i/(e_1 + e_2)$.
3. A constant $\lambda \geq 1$ sums the marginal costs of player i's expenses and of any additional destruction of the pre-war resources; the marginal cost of such destruction is $\lambda - 1$. Formally, $k_i = 1$, $w_{ii} = 1 - \lambda$, $w_{ij} = -\lambda$, $l_{ii} = 1$ and $l_{ij} = 0$.

[12]For example, The Art of War by Sun Tzu, dating back to the fifth century BC, recommends "[i]f your opponent is temperamental, seek to irritate him. Pretend to be weak, that he may grow arrogant." Joachim Peiper, a famous World War II military commander, was also famously optimistic.

[13]In the special case of $\mu_1 = \mu_2$, the success function defined by (10) takes the relative-difference form that Beviá and Corchón (2015) proposed to combine the desirable properties of ratio-form and difference-form success functions. Assuming $\mu_1 = \mu_2$, function (10) is also similar to Nitzan's (1991) sharing rule in collective contests and the success function that Balart et al. (2017) proposed to use in individual contests. Moreover, assuming $\mu_1 = \mu_2 = 0.5$, function (10) specializes to the one that Plott (1987) applied to study cost-shifting rules in civil litigation. However, while function (10) is confined to two players, the sharing rule in Nitzan (1991) and the success function in Beviá and Corchón (2015) and Balart et al. (2017) can be extended to more than two players.

4. Player i's payoff function U_i is given by

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)\big[v - \lambda e_i - \lambda e_j\big].$$

In the Conquest Game, the players simultaneously spend e_1, e_2 to determine who will obtain their combined pre-war resources less the dissipation arising from their military expenses and any additional destruction: $v - \lambda e_1 - \lambda e_2$. Being conquered by the winner, the loser obtains no resources and loses all her remaining resources.[14]

The specifications in part 3 serves the following functions. First, the specification $k_i = 1$ ensures the players' military expenses e_i, e_j to be equally effective in producing spillovers to themselves and to each other; this specification removes cross-player asymmetries in producing spillovers. Secondly, the specifications $w_{ii} = 1 - \lambda$ and $l_{ii} = 1$ together simplify calculation and presentation by letting one parameter — λ — to capture how e_i affects player i's prize of conquest in the event of her victory. Similarly, the specifications $w_{ij} = -\lambda$ and $l_{ij} = 0$ together simplify calculation and presentation by letting λ also to capture how e_j affects player i's prize of conquest in the event of her victory.

4.2 Equilibrium

Corollary 2 characterizes the unique nontrivial equilibrium of the Conquest Game.

Corollary 2. *The Conquest Game has a unique nontrivial Nash equilibrium* $\left(e_1^*, e_2^*\right)$, *which is characterized by*

$$e_1^* = \frac{v(1-\eta)(1-\eta+\eta\mu_2)}{\lambda[2(1-\eta)+\eta(\mu_1+\mu_2)]^2}, \qquad e_2^* = \frac{v(1-\eta)(1-\eta+\eta\mu_1)}{\lambda[2(1-\eta)+\eta(\mu_1+\mu_2)]^2}.$$

In equilibrium, players 1 *and* 2 *respectively believe that their probabilities of winning are* θ_1^*, θ_2^* *given by*

$$\theta_1^* = \eta\mu_1 + \frac{(1-\eta)(1-\eta+\eta\mu_2)}{2(1-\eta)+\eta(\mu_1+\mu_2)}, \qquad \theta_2^* = \eta\mu_2 + \frac{(1-\eta)(1-\eta+\eta\mu_1)}{2(1-\eta)+\eta(\mu_1+\mu_2)}.$$

Corollary 2 reveals that the equilibrium expenses differ to the extent of the relative advantages parameters μ_1, μ_2 in the numerator. In the special case of $\mu_1 = \mu_2$, the players' equilibrium expenses are equal. The remainder of this subsection will analyze the equilibrium implications of changes in μ_1, μ_2.

[14]For example, this specification captures the Norman conquest of England in 1066, as well as the Mongol conquests in the thirteenth century.

Intuitively, the parameter μ_i measures player i's individual tastes for probabilities. When μ_i increases while the pair of expenses (not necessarily the equilibrium pair) is fixed, player i believes that her probability of winning increases. For this reason, call player i **relatively more optimistic ex ante** if $\mu_i > \mu_j$. Conversely, call player i **relatively more pessimistic ex ante** if $\mu_i < \mu_j$.

Corollary 3. *Consider the nontrivial Nash equilibrium of the Conquest Game.*

1. *The player who is relatively more optimistic (respectively, pessimistic) ex ante spends relatively less (more). Formally, $e_i^* > e_j^*$ if and only if $\mu_i < \mu_j$.*
2. *The player who is relatively more optimistic (respectively, pessimistic) ex ante believes in a relatively greater (smaller) probability of winning. Formally, $\theta_i^* < \theta_j^*$ if and only if $\mu_i < \mu_j$.*

Part 1 of Corollary 3 reveals that the player who is relatively more pessimistic ex ante incurs relatively *more* expenses in equilibrium. Intuitively, she does so to offset her relative disadvantages. Part 2 reveals that her relatively more expenses only partially offsets her relative disadvantages; she still believes in a smaller probability of winning compared to her opponent. The converse is true for the player who is relatively more optimistic ex ante.

Corollary 3 adds to the existing analyses of asymmetries in military spending. In particular, using a Tullock success function, Skaperdas and Syropoulos (1997) constructed a two-player model in which each player allocates her resources between military spending and useful production (see Garfinkel and Skaperdas 2007 for a simplified version). One of their equilibrium results is that the player who is relatively less effective in useful production tends to spend relatively more on her military; she is thus more likely to win (Skaperdas and Syropoulos 1997 pp. 105-106, Garfinkel and Skaperdas 2007 pp. 665-667). Unlike their model, the present Conquest Game does not account for differences in productiveness, but captures differences in individual tastes for probabilities with a different success function (see (10)). Without allowing for differences in productiveness, Corollary 3 nonetheless reveals optimism and pessimism ex ante as a source of asymmetry in military spending.

To measure the players' collective tastes for probabilities, define the **degree of collective optimism** as $\sigma = \mu_1 + \mu_2 - 1$. Some algebra using equation (10) obtains

$$\theta_1(e_1, e_2) + \theta_2(e_2, e_1) - 1 = \eta\sigma, \tag{11}$$

for all pairs of positive expenses, not just the equilibrium pair. The value of $\theta_1(e_1, e_2) + \theta_2(e_2, e_1)$ sums the players' respective probabilities of winning according to their own beliefs; $\theta_1(e_1, e_2) + \theta_2(e_2, e_1) = 1$ if and only if these beliefs coincide.

A non-zero degree captures collective optimism or pessimism. When $\sigma > 0$, equation (11) reveals that the probabilities of winning sum to $\theta_1 + \theta_2 > 1$; this captures the scenario of the players being **collectively optimistic**. When $\sigma < 0$, the probabilities of winning sum to $\theta_1 + \theta_2 < 1$; this captures the scenario of the players being **collectively pessimistic**. Because the sum of the players' probabilities of winning is increasing with σ, the players become collectively more optimistic (respectively, collectively more pessimistic) if σ increases (decreases).

Corollary 4. *In the nontrivial Nash equilibrium of the Conquest Game, the players collectively make more (respectively, less) expenses if their degree of optimism σ decreases (increases). Formally,*

$$\frac{d}{d\sigma}\left(e_1^* + e_2^*\right) < 0.$$

A surprising result, Corollary 4 reveals that more collective optimism (respectively, pessimism) decreases (increases) total expenses in equilibrium. Intuitively, as the players become collectively more optimistic, they believe that they can incur less expenses to offset the influence of their relative advantages. The converse is true if the players become collectively more pessimistic.

The result that total equilibrium expenses change in response to optimism or pessimism affects the players' total equilibrium payoffs ex ante, denoted U^*, where[15]

$$U^* = U_1\left(e_1^*, e_2^*\right) + U_2\left(e_2^*, e_1^*\right) = \left(\theta_1^* + \theta_2^*\right)\left[v - \lambda\left(e_1^* + e_2^*\right)\right]. \tag{12}$$

Corollary 5. *In the nontrivial Nash equilibrium of the Conquest Game, as the players' degree of optimism increases, the sum of their ex-ante payoffs increases. Formally,*

$$\frac{dU^*}{d\sigma} > 0.$$

Corollary 5 reflects the direct and indirect effects that collective optimism σ has on total ex-ante payoffs U^*. Directly, a greater σ increases U^* by raising the players' total probabilities of success in equilibrium, $\theta_1^* + \theta_2^*$, according to their own beliefs (see equation (11)). Indirectly, a greater σ decreases the players' total expenses in equilibrium (see Corollary 4); indicating a smaller dissipation of resources arising from military expenses, this indirect effect increases U^*. Thus both the direct and indirect effects of an increase in σ increase U^*.

[15]The value U^* is an ex-ante notion because it sums the players' equilibrium payoffs in expectation, before the winner of the war is determined. After the winner is determined, her ex post payoff is $v - \lambda\left(e_1^* + e_2^*\right)$ while the loser's is 0. The sum of payoffs ex-post is thus $v - \lambda\left(e_1^* + e_2^*\right)$, which is smaller than (respectively, equal to, greater than) U^* if and only if the degree of optimism $\sigma > 0$ ($= 0$, < 0).

5 Application: R&D Spillovers with Asymmetric Advantages

Returning to the standard case of probabilities of success summing to 1, this section applies the Contest Game to model spillovers in R&D. For reasons including information sharing and imperfect protection of intellectual property, firms may benefit from each other's R&D expenses. D'Aspremont and Jacquemin (1988), Kamien et al. (1992), and Hartwick (1984) were among the first to offer economic analysis of these positive spillovers (or externalities). Using a Tullock contest model with multiple symmetric players, Chung (1996) captured R&D spillovers with a concave prize function that is increasing with total expenses. More recently, Chowdhury and Sheremeta (2011a) constructed a Tullock contest model with two symmetric players and a prize function that is linearly separable in each player's expenses; their model captured R&D spillovers by specifying that the prize is increasing with each player's expenses. Building upon their effort, the following will reveal the implications of R&D spillovers when the players have asymmetric advantages.

5.1 Setup

Construct the **R&D Game** by specializing the following features of the Contest Game:

1. The players value the (non-spillover) prize of winning equally; $v_1 = v_2 = v$ for a constant $v > 0$.
2. Player 1's success function θ_1 has the following Tullock form:

$$\theta_1(e_1, e_2) = \begin{cases} \frac{\mu e_1}{\mu e_1 + (1-\mu) e_2} & \text{if } e_1 + e_2 > 0, \\ \mu & \text{otherwise,} \end{cases} \tag{13}$$

   where the constant $\mu \in (0, 1)$ captures player 1's relative advantages in R&D.[16]
3. Player 2's success function is $\theta_2 = 1 - \theta_1$. The value $1 - \mu$ captures her relative advantages.
4. Player i's own expenses give rise to no spillovers to herself; $w_{ii} = l_{ii} = 0$.
5. The players linearly generate the same non-negative spillovers to each other, where these spillovers satisfy Assumption 3. Formally, $k_i = 1$, $w_{ij} = w$ and $l_{ij} = l$ for constants $w, l \geq 0$ satisfying $w - l < \min\{\mu^{-1}(1-\mu), (1-\mu)^{-1}\mu\}$.
6. Player i's payoff function U_i is

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)\big[v + w e_j\big] + \big[1 - \theta_i(e_i, e_j)\big] l e_j - e_i. \tag{14}$$

[16]The special case of $\mu = 0.5$ captures the success function in the two-player Tullock model with positive spillovers that Chowdhury and Sheremeta (2011a) (at p. 418) constructed.

In the R&D Game, the players simultaneously make R&D expenses e_1, e_2 to compete for the prize v. For instance, the prize may be a successful patent. The players' expenses generate non-negative spillovers w, l to each other; the winner and loser receive different spillovers if $w \neq l$. The players are symmetric except in respect of their relative advantages in R&D, measured by μ. Player 1's (respectively, player 2's) probability of winning increases (decreases) with μ.

The specifications in part 5 serves the following functions. First, the specification $k_i = 1$ ensures the players' R&D expenses e_i, e_j to be equally effective in producing R&D spillovers to each other; this specification removes cross-player asymmetries in producing spillovers to each other. Secondly, the specifications $w_{ij} = w$ and $l_{ij} = l$ together simplify presentation by removing the subscript ij, which, in more general settings, captures cross-player differences in producing spillovers to each other (see section 2). Thirdly, as an implication of Assumption 3, the specification $w - l < \min\{\mu^{-1}(1-\mu), (1-\mu)^{-1}\mu\}$ ensures that equilibrium expenses are positive and bounded above (see Corollary 6 in subsection 5.2). This specification requires that the winner's spillovers w be not too much greater than the loser's spillovers l, in order to prevent differences in spillovers to incentivize explosive R&D expenses.

5.2 Equilibrium

Corollary 6 provides a closed-form expression for the unique nontrivial equilibrium of the R&D Game. The rest of this subsection will analyze the properties of this equilibrium.

Corollary 6. *The R&D Game has a unique nontrivial Nash equilibrium $\left(e_1^*, e_2^*\right)$, which is characterized by*

$$e_1^* = e_2^* = \frac{v\mu(1-\mu)}{1-(w-l)\mu(1-\mu)}.$$

In equilibrium, players 1 and 2's probabilities of winning are respectively $\theta_1^* = \mu$, $\theta_2^* = 1-\mu$.

Let $\delta = w - l$ denote the (cross-player) spillover differential. Our specification ensures that δ is bounded above by $\min\{\mu^{-1}(1-\mu), (1-\mu)^{-1}\mu\}$, and does not impose a lower bound; δ potentially may be negative. To player i, δ interacts with her opponent's effort e_j. Suppose e_j increases by one unit. Then δ is the spillover premium that player i receives from winning rather than losing. In other words, δ is the per-unit spillover gain of winning. Thus δ is a part of the stakes of the contest.

Let $\tau = \mu(1-\mu)$ measure the balancedness of the players' relative advantages; τ increases (respectively, decreases) if their relative advantages become more balanced (extreme), that is, $|\mu - 0.5|$ decreases (increases).

Corollary 7 considers how changes in δ and τ affect equilibrium expenses in the R&D Game.

Corollary 7. *In the R&D Game, a greater spillover differential increases total expenses in equilibrium. Formally,*

$$\frac{d}{d\delta}\left(e_1^* + e_2^*\right) > 0. \tag{15}$$

Moreover, when the relative advantages of the players become more extreme (that is, τ decreases), the magnitude of the resulting increase in total equilibrium expenses becomes smaller, and diminishes in the limit. Formally,

$$\frac{d}{d\tau}\left(\frac{d}{d\delta}\left(e_1^* + e_2^*\right)\right) > 0, \tag{16}$$

$$\lim_{\tau \to 0+}\left(\frac{d}{d\delta}\left(e_1^* + e_2^*\right)\right) = 0. \tag{17}$$

Inequality (15) in Corollary 7 first reveals that a greater spillover differential δ — for instance, due to a player receiving *smaller* loser's spillovers (l decreases) or greater winner's spillovers (w increases) from her *opponent* — heightens the players' collective incentives to spend on R&D. Chowdhury and Sheremeta (2011a) (at p. 418) found such a result in a symmetric Tullock contest, and the present Corollary 7 confirms the result continues to hold in the presence of asymmetric advantages.

To see the intuition underlying inequality (15), rewrite player i's payoff function in equation (14) as

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)\left[v + \delta e_j\right] - e_i + l e_j, \tag{18}$$

where the component $v + \delta e_j$ sums player i's total marginal benefits of winning rather than losing the R&D contest; δ is a part of that component. Suppose δ increases. In response to the resulting increase in the stakes of the contest, player i has incentives to increase her R&D expenses, e_i. A greater δ and a greater e_i then increase the stake of the contest from player j's perspective, heightening her incentives to spend as well. As inequality (15) confirms, this chain reaction continues until the players reach an equilibrium with greater collective expenses than before the increase in δ.

Now consider to the case of asymmetric advantages ($\mu \neq 0.5$). Inequality (16) in Corollary 7 reveals that when the players' relative advantages become more extreme (τ decreases), changes in the spillover differential δ have a smaller impact on incentives to spend on R&D. Measuring how δ affects total equilibrium expenses at the margin, the value of the derivative $\frac{d}{d\delta}\left(e_1^* + e_2^*\right)$ decreases when τ decreases, according to inequality (16). Thus inequality (16) reveals that changes in δ have a smaller impact on R&D expenses as τ decreases. Indeed, equation (17) further reveals that in the limit when τ approaches 0 — indicating an extreme contest in which one player has absolute advantages over her opponent — the value of $\frac{d}{d\delta}\left(e_1^* + e_2^*\right)$ approaches zero. In other words, as the contest becomes very imbalanced, changes in δ have a very small impact on incentives to spend on R&D.

Corollary 7 has policy implications. Suppose a social planner wishes to increase R&D spending. Intellectual property law is among the legal mechanisms that she may use to alter the extent of R&D spillovers. For example, a race to invent a new product is a R&D contest, and strengthening patent protection can reduce the extent to which the loser can profit from the winner's invention. In the language of the present R&D Game, smaller loser's spillovers (l decreases) reflect strengthened protection of patents (or other intellectual properties). Ceteris paribus, a smaller l leads to a greater spillover differential δ. This results in greater R&D expenses in equilibrium, as inequality (15) in Corollary 7 reveals and the above discussion explains. Corollary 7 thus offers guidance on how to affect incentives to spend on R&D.

Moreover, Corollary 7 suggests the *effectiveness* of policy interventions to affect R&D spending depends on the relative advantages of the players involved. As inequality (16) reveals and the above discussion explains, when the relative advantages of the players become more balanced (τ increases), changes in the spillover differential δ have a greater impact on incentives to spend. Changes in δ capture policy interventions to alter R&D spillovers. For example, strengthening intellectual property protection can decrease the loser's spillovers l or increase the winner's spillovers w. Thus, in terms of affecting incentives to spend on R&D, policy interventions that change R&D spillovers are more effective in scenarios involving R&D firms that are similarly competitive than in scenarios involving a dominant firm.

Let U_i^* denote player i's payoff in equilibrium. Corollary 8 considers how changes in parameters affect the players' total payoffs in equilibrium, $U_1^* + U_2^*$. Their total payoffs do not account for the public benefits of R&D.

Corollary 8. *Consider the equilibrium of the R&D Game.*

1. *If the winner's spillovers increase, then total payoffs increase. Formally,*

$$\frac{d}{dw}\left(U_1^* + U_2^*\right) > 0.$$

2. *Suppose the winner's spillovers are sufficiently small (respectively, large). Then as the loser's spillovers increase, total payoffs increase (decrease). Formally,*

$$w < \frac{1 + 2\mu(1-\mu)}{2\mu(1-\mu)} \qquad \Rightarrow \qquad \frac{d}{dl}\left(U_1^* + U_2^*\right) > 0$$

$$w = \frac{1 + 2\mu(1-\mu)}{2\mu(1-\mu)} \qquad \Rightarrow \qquad \frac{d}{dl}\left(U_1^* + U_2^*\right) = 0$$

$$w > \frac{1 + 2\mu(1-\mu)}{2\mu(1-\mu)} \qquad \Rightarrow \qquad \frac{d}{dl}\left(U_1^* + U_2^*\right) < 0.$$

3. *Suppose total spillovers are sufficiently small (respectively, large). Then as the relative advantages of the players become more balanced, total payoffs decrease (increase). Formally,*

$$w + l < 2 \qquad \Rightarrow \qquad \frac{d}{d\tau}\left(U_1^* + U_2^*\right) < 0$$

$$w + l = 2 \qquad \Rightarrow \qquad \frac{d}{d\tau}\left(U_1^* + U_2^*\right) = 0$$

$$w + l > 2 \qquad \Rightarrow \qquad \frac{d}{d\tau}\left(U_1^* + U_2^*\right) > 0.$$

Part 1 of Corollary 8 reveals that more (cross-player) winner's spillovers (w) lead to greater total payoffs in equilibrium, while part 2 reveals that how changes in (cross-player) loser's spillovers (l) affect total equilibrium payoffs depend on w; if w is small (respectively, large) in the sense described in part 2, then total payoffs in equilibrium increase (decrease) with l.

Part 3 of Corollary 8 reveals that how changes in the players' relative advantages affect their total payoffs in equilibrium depend on total (cross-player) spillovers, $w + l$. If $w + l$ is sufficiently small (respectively, large) in the sense described in part 3, then total payoffs in equilibrium decrease (increase) as the relative advantages become more balanced, that is, τ increases.

5.3 Robustness of the Tullock Success Function

A conventional wisdom in the literature on Tullock contest models is that as the relative advantages of the players become more balanced, their total expenses increase.[17] To highlight the importance of robust formulation of success functions, this subsection reveals that the conventional wisdom has limitations.

First, Corollary 9 confirms that the conventional wisdom holds in the R&D Game, which adopts the Tullock success function.

Corollary 9. *In the R&D Game, as the relative advantages of the players become more balanced,*

[17]For such a result in a Tullock contest with asymmetric technologies, see Cornes and Hartley (2005) (pp. 940-41). For such a result in a model of litigation, see Carbonara et al. (2015) (pp. 6-7). For a recent paper that shows this result may not hold in a general model of litigation, see Chen and Rodrigues-Neto (2017).

their total expenses in equilibrium increase. Formally,

$$\frac{d}{d\tau}\left(e_1^* + e_2^*\right) > 0.$$

Corollary 9 reveals that the players have heightened incentives to spend on R&D as their relative advantages, measured by τ, become more balanced.

Now construct an **Alternative R&D Game** by modifying the R&D Game only in respect of the success functions. In the Alternative R&D Game, let players 1 and 2's respective success functions θ_1, θ_2 take the following forms:

$$\theta_1(e_1, e_2) = \begin{cases} \eta\mu + (1-\eta)\frac{e_1}{e_1+e_2} & \text{if } e_1 + e_2 > 0, \\ \mu & \text{otherwise,} \end{cases} \tag{19}$$

$$\theta_2(e_2, e_1) = 1 - \theta_1(e_1, e_2),$$

where constants $\mu, \eta \in (0, 1)$ are respectively player 1's relative advantages in R&D, and the weight that she assigns to μ rather than to her share of total expenses, $e_1/(e_1 + e_2)$. As some algebra will reveal, the value $1 - \mu$ captures player 2's relative advantages, and η is the weight that she assigns to $1 - \mu$ rather than to her share of total expenses, $e_2/(e_1 + e_2)$.

In the Alternative R&D Game, player i believes that i's probability of winning is what her opponent j believes to be j's probability of losing; formally, $\theta_i = 1 - \theta_j$. Thus, unlike the Conquest Game (see section 4), the players in the Alternative R&D Game are neither collectively optimistic nor collectively pessimistic. Instead, the success function adopted in the Alternative R&D Game captures the intuition that a player's probability of winning the R&D contest can be the weighted average of her relative advantages and her share of total expenses.

Corollary 10 finds and characterizes the unique nontrivial Nash equilibrium of the Alternative R&D Game.

Corollary 10. *The Alternative R&D Game has a unique nontrivial Nash equilibrium* $\left(e_1^*, e_2^*\right)$*, which is characterized by*

$$e_1^* = e_2^* = \frac{(1-\eta)v}{4 - (1-\eta)\delta}.$$

In equilibrium, players 1 *and* 2*'s probabilities of winning are respectively* $\theta_1^* = (\mu - 0.5)\eta + 0.5$*,* $\theta_2^* = (0.5 - \mu)\eta + 0.5$.

Using the close-form expression for the nontrivial Nash equilibrium of the Alternative R&D Game, Corollary 11 ascertains how total expenses in equilibrium respond to changes in the relative advantages of the players, and in the spillovers.

Corollary 11. *Consider the equilibrium of the Alternative R&D Game.*

1. *As the relative advantages of the players change, their total expenses remain the same. Formally,*

$$\frac{d}{d\tau}\left(e_1^* + e_2^*\right) = 0.$$

2. *As the spillover differential increases, total expenses increase. Formally,*

$$\frac{d}{d\delta}\left(e_1^* + e_2^*\right) > 0. \tag{20}$$

A comparison of Corollary 9 (for the original R&D Game) and the present Corollary 11 first reveals that, due to the adoption in the Alternative R&D Game of the success function (19), changes in the players' relative advantages *no longer* affect their total expenses in equilibrium. Thus the conventional wisdom regarding Tullock success functions does *not* apply to the Alternative R&D Game; in this Game, when the contest becomes more balanced (τ increases), total expenses do not change. This result identifies a limitation of Tullock success functions, and highlights the importance of inquiries into the robustness of contest models.

Moreover, a comparison of inequality (15) in Corollary 7 (for the original R&D Game) and inequality (20) in Corollary 11 (for the Alternative R&D Game) suggests that comparative statics regarding the spillover differential δ are robust. Under the Tullock success function adopted in the original R&D Game and under the success function (19) adopted in the Alternative R&D Game, a greater δ heightens incentives to spend on R&D. This result suggests that equilibrium predictions and policy recommendations regarding δ are robust to different formulations of success functions.

6 Remarks on Generality

6.1 Uncertain Success Functions

This subsection will illustrate that the Contest Game captures uncertainty regarding the success functions. The approach taken is to establish equivalence conditions, which approach is in the same spirit as Baye and Hoppe (2003) and Chowdhury and Sheremeta (2015).

Construct the **Uncertain Contest Game** by making the following modifications to the Contest Game:

1. At the time of making expenses, player i is uncertain as to which one of a finite collection of $N \geq 1$ success functions $\{\theta_{i(1)}, \theta_{i(2)}, ...\theta_{i(N)}\}$ will determine her probability of winning. Each $\theta_{i(z)}$, $z \in \{1, ..., N\}$, satisfies Assumptions 1-6.
2. It is common knowledge that player i assigns the prior probability $p_{i(z)} \geq 0$ to $\theta_{i(z)}$ being operative, where $\sum_{z=1}^{n} p_{i(z)} = 1$.
3. Player i's payoff function is $\widetilde{U}_i : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\widetilde{U}_i(e_i, e_j) = \sum_{z=1}^{n} p_{i(z)} \Big(\theta_{i(z)}(e_i, e_j) \Big[v_i + w_{ii} e_i + w_{ij} e_j^{k_i} \Big] + \big[1 - \theta_{i(z)}(e_i, e_j) \big] \Big[l_{ii} e_i + l_{ij} e_j^{k_i} \Big] - e_i \Big). \tag{21}$$

Player i's payoff in the present Uncertain Contest Game is the weighted average of her expected monetary outcome in the original Contest Game; the weights are her prior probabilities regarding the operative success function. She assigns the prior probability $p_{i(z)}$ to the event that the success function $\theta_{i(z)}$ will operate to determine her probability of winning. In that event, she obtains her expected monetary outcome in the Contest Game given the success function $\theta_{i(z)}$.

The remainder of this subsection will reveal that the present Uncertain Contest Game is strategically equivalent to the original Contest Game. To see this, construct a success function θ_i by

$$\theta_i(e_i, e_j) = \sum_{z=1}^{n} p_{i(z)} \theta_{i(z)}(e_i, e_j). \tag{22}$$

Some algebra comparing player i's payoff U_i (defined by (1)) in the Contest Game and her payoff $\widetilde{U}_i$ (defined by (21)) in the Uncertain Contest Game reveals that these payoffs are equal:

$$\widetilde{U}_i(e_i, e_j) = \sum_{z=1}^{n} p_{i(z)} \theta_{i(z)}(e_i, e_j) \Big[v_i + w_{ii} e_i + w_{ij} e_j^{k_i} \Big] + \left(1 - \sum_{z=1}^{n} p_{i(z)} \theta_{i(z)}(e_i, e_j) \right) \Big[l_{ii} e_i + l_{ij} e_j^{k_i} \Big] - e_i$$

$$\widetilde{U}_i(e_i, e_j) = \theta_i(e_i, e_j) \Big[v_i + w_{ii} e_i + w_{ij} e_j^{k_i} \Big] + \big[1 - \theta_i(e_i, e_j) \big] \Big[l_{ii} e_i + l_{ij} e_j^{k_i} \Big] - e_i = U_i. \tag{23}$$

Proposition 2 will establish that the success function θ_i defined by (22) satisfies Assumptions 1-6.

Proposition 2. *Consider a finite collection of success functions $\{\theta_{i(1)}, \theta_{i(2)}, ...\theta_{i(N)}\}$. If each one of them satisfies Assumptions 1-6, then their convex combination also satisfies Assumptions 1-6.*

An application of Proposition 2 reveals that adopting for the original Contest Game the success function θ_i defined by (22) satisfies Assumptions 1-6. Hence Proposition 1 finds and characterizes a nontrivial Nash equilibrium in the Contest Game given the success function θ_i defined by (22). This equilibrium is also a nontrivial Nash equilibrium in the Uncertain Contest Game, due to the equality of payoffs established in (23).

Proposition 2 thus reveals that the original Contest Game captures not only a large class of success functions that appear in the contest theory literature, but also captures the class of convex combinations of these success functions as well. This attests to the generality and robustness of the original Contest Game and of its equilibrium predictions and normative implications.

6.2 Homogeneous Expenses and Spillovers

This subsection will illustrate that the Contest Game captures expenses and spillovers that are homogeneous functions.

Construct the **Homogeneous Contest Game** by modifying the Contest Game as follows:

1. Let $y_i \geq 0$ be player i's choice variable, which is interpreted as her **effort**. A strictly increasing homogeneous function $E_i : \mathbb{R}_+ \to \mathbb{R}_+$ of degree $k_{ii} > 0$ gives $E_i(y_i)$ as her **costs** of exerting effort.[18] Let $\theta_i(y_i, y_j)$ denote what she believes to be her probability of winning when she and her opponent j respectively exert y_i, y_j levels of efforts.
2. Homogeneous functions $W_{ii}, L_{ii} : \mathbb{R}_+ \to \mathbb{R}$ of degree k_{ii} respectively give the winner's and loser's spillovers of player i arising from her own effort y_i; W_{ii}, L_{ii} have the same degree of homogeneity as E_i. Let the function $\Delta_{ii} = W_{ii} - L_{ii}$ capture the spillover differential arising from y_i and affecting player i.
3. Homogeneous functions $W_{ij}, L_{ij} : \mathbb{R}_+ \to \mathbb{R}$ of degree $k_{ij} > 0$ respectively give the winner's spillovers and loser's spillovers of player i arising from her opponent j's effort y_j; these functions may not have the the same degree of homogeneity as player i's cost function E_i or her opponent j's cost function E_j. Let the function $\Delta_{ij} = W_{ij} - L_{ij}$ capture the spillover differential arising from y_j and affecting player i.
4. Player i's payoff is $\widehat{U}_i : \mathbb{R}^2_+ \to \mathbb{R}$ given by

$$\widehat{U}_i(y_i, y_j) = \theta_i(y_i, y_j)\big[v_i + W_{ii}(y_i) + W_{ij}(y_j)\big] + \big[1 - \theta_i(y_i, y_j)\big]\big[L_{ii}(y_i) + L_{ij}(y_j)\big] - E_i(y_i). \tag{24}$$

Proposition 3 proves that a change of variables reveals the present Homogeneous Contest Game is strategically equivalent to the original Contest Game.[19]

[18] We assume the fixed cost of exerting effort is zero. For a discussion of the relevance of fixed costs, see Remark 1 in Chen and Rodrigues-Neto (2017).

[19] Chowdhury and Sheremeta (2015) offered strategic equivalence conditions and results among two-player symmet-

Proposition 3. *Suppose the spillover parameters in the original Contest Game are*

$$w_{ii} = \frac{W_{ii}(1)}{E_i(1)}, \qquad l_{ii} = \frac{L_{ii}(1)}{E_i(1)}, \qquad w_{ij} = \frac{W_{ij}(1)}{E_j(1)^{\frac{k_{ij}}{k_{jj}}}}, \qquad l_{ij} = \frac{L_{ij}(1)}{E_j(1)^{\frac{k_{ij}}{k_{jj}}}}, \qquad k_i = \frac{k_{ij}}{k_{jj}}. \tag{25}$$

Then a pair of positive efforts (y_1^*, y_2^*) is a nontrivial Nash equilibrium of the Homogeneous Contest Game if and only if it relates to a nontrivial Nash equilibrium (e_1^*, e_2^*) of the original Contest Game via the following transformation:

$$\left(y_1^*, y_2^*\right) = \left(\left(\frac{e_1^*}{E_1(1)}\right)^{\frac{1}{k_{11}}}, \left(\frac{e_2^*}{E_2(1)}\right)^{\frac{1}{k_{22}}}\right). \tag{26}$$

Equation (26) in Proposition 3 characterizes a bijection between a nontrivial Nash equilibrium of the Homogeneous Contest Game ((y_1^*, y_2^*)) and a nontrivial Nash equilibrium of the original Contest Game ((e_1^*, e_2^*)). Intuitively, player i faces the same incentives for choosing expenses e_i in the original Contest Game and choosing effort y_i^* in the Homogeneous Contest Game; equation (26) implies $e_i^* = E_i(y_i^*)$. The original Contest Game thus captures the players' incentives when their cost functions and spillover functions are homogeneous as well. This result further attests to the generality and robustness of the original Contest Game and of its equilibrium properties.

6.3 Relationship with Tullock Contests

This subsection considers the extent to which the Contest Game generalizes contest models based on the Tullock success function.

Consider a success function $\rho_i : \mathbb{R}_+^n \to [0,1]$ that gives $\rho_i(e)$ as the probability of success of player $i \in \{1, ..., n\}$, $n > 1$, where $e = (e_1, ..., e_n)$ is the vector of strategies. The following Assumption 7 restricts ρ.

Assumption 7. *The success function ρ satisfies the following properties:*

1. $1 > \rho_i(e) \geq 0$ *and* $\sum_i \rho_i(e) = 1$*; if* $e_i > 0$ *then* $\rho_i(e) > 0$.
2. $\rho_i(e)$ *is strictly increasing in* e_i *and nonincreasing in* e_j, $j \neq i$.
3. *(Independence of irrelevant alternatives.)* $\rho_i(e_1, ..., e_{k-1}, 0, e_{k+1}, ..., e_n) = \frac{\rho_i(e)}{1-\rho_k(e)}$ *for every* $i \neq k$.

ric Tullock contests with linear cost and spillover functions. Building upon their efforts, the present section 6.2 presents equivalence conditions and results that allow for non-specified success functions, homogeneous cost and spillover functions, and asymmetries therein.

4. *(Homogeneity of degree zero.)* $\rho_i(e) = \rho_i(xe)$ *for every* i*, every* $x > 0$ *where* $xe = (xe_1, ..., xe_i, ..., xe_n)$*.*

Building on the seminal work of Skaperdas (1996), Clark and Riis (1998) proved that Assumption 7 holds if and only if ρ takes the following asymmetric Tullock form:

$$\rho_i(e) = \frac{z_i e_i^\gamma}{\sum_{j=1}^n z_j e_j^\gamma},$$

where constants $\gamma, z_i, z_j > 0$.

Because the present Contest Game is a two-player model, it does not generalize n-player contest models based on Assumption 7. By comparison, the Contest Game permits individualized success functions (see section 4) and differences in returns to expenses (see Assumption 1), while these behavioral traits respectively violate properties 1 and 4 of Assumption 7.

Confined to the scenario of two players sharing the same success function, having equal returns to expenses and generating zero spillovers, Assumption 7 is a special case of Assumptions 1-6 of the Contest Game. The opposite is not true. To see this, consider the success function of the Conquest Game (see equation (10) in section 4), which falls within the scope of Assumptions 1-6. If $\mu_1 = \mu_2$, $e_1 > 0$ and $e_2 = 0$, then player 1's probability of winning is

$$\theta_1(e_1, 0) = 1 - (1 - \mu_1)\eta,$$

which violates property 3 of Assumption 7, because $\theta_1(e_1, 0) < 1 = \theta_1(e_1, e_2)/[1 - \theta_2(e_2, e_1)]$ for any $e_2 > 0$.

7 Discussion

Extending the descriptive scope of contest theory, the present Contest Game allows for general and individualized success functions and spillovers. Future research may proceed in several directions.

First, the Contest Game is a two-player model; future research may introduce more than two players. Although two-player contests are common, many real-life contests — such as the U.S. presidential primaries and multi-state wars — have more than two contestants. Group contests such as team sports also have more than two players (see Serena and Corchón's 2017 survey, pp. 25-27). It may be fruitful to explore the implications of general and individualized success functions and spillovers in multi-player contests.

Secondly, extending the Contest Game to multiple periods may capture intertemporal incentives and information revelation over time. These factors have implications in military contests (see

Garfinkel and Skaperdas 2000 and Corchón and Yıldızparlak 2013). Many other real-life scenarios also answer the description of dynamic contests (see Serena and Corchón's 2017 survey, pp. 20-23); elimination tournaments and sports leagues are among the prominent examples. Giving a dynamic structure to the Contest Game may reveal how intertemporal incentives and information revelation over time may affect the differences in the players' beliefs regarding the success functions.

Thirdly, future research may modify the Contest Game to incorporate private information. The players in the present Contest Game may have non-common beliefs regarding the success functions, but their common knowledge includes their payoff functions and the parameters characterizing their spillovers. It may be fruitful to explore the role of spillovers by building upon models that assume common priors but permit private information. For example, Einy et al. (2015) proved the existence of a pure-strategy Bayesian-Nash equilibrium in Tullock contests with private information.

Fourthly, future research may consider modifying the Contest Game to account for the possibility of a draw. Generalizing Skaperdas (1996) and Clark and Riis (1998), Blavatskyy (2010) axiomatized a Tullock contest that permits a draw. Chowdhury (2017) characterized equilibria for an all-pay auction in which the highest bid may fail to win the prize, and the prize value may depend on the bid level. Many real-life contests, such as individual matches in the group stage of the FIFA World Cup, can result in no winner. Modifying the Contest Game to permit a draw may capture these contests without specifying the functional form of the success function.

Finally, future research may build upon the present efforts to develop contest models that capture well-documented behavioral traits. Optimism and pessimism are prominent behavioral traits that fall within the scope of the present Contest Game (see section 4). However, the present assumptions do not aim to capture some behavioral traits that consistently appear in contest experiments, such as preferences for relative outcomes (see Dechenaux et al.'s 2015 survey at pp. 614-616). Future research may modify the Contest Game to incorporate these behavioral traits.

A Appendix: Proofs

This appendix contains all proofs.

Lemma 4 is a technical lemma that will facilitate calculations. For each $i \in \{1,2\}$, define functions $\theta_i^{(j)}, \phi_{ii} : \mathbb{R}_{++} \to \mathbb{R}$ by

$$\theta_i^{(j)}(r_{ij}) = \frac{\partial}{\partial r_{ij}} \theta_i\left(r_{ij}^{-1}\right) \tag{27}$$

$$\phi_{ii}(r_{ij}) = \frac{-v_i \theta_i^{(j)}(r_{ij})}{1 - l_{ii} - \delta_{ii}\theta_i\left(r_{ij}^{-1}\right) + \left(\delta_{ii} + \delta_{ij} r_{ij}\right) r_{ij} \theta_i^{(j)}(r_{ij})}. \tag{28}$$

Lemma 4. *Whenever expenses $e_i, e_j > 0$ for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, the following properties hold:*

1. *The expenses ratios r_{ii}, r_{ij} and success function θ_i satisfy*

$$\frac{\partial r_{ii}}{\partial e_i} = \frac{r_{ii}}{e_i}, \quad \frac{\partial r_{ij}}{\partial e_i} = \frac{-r_{ij}}{e_i}, \quad \frac{\partial \theta_i}{\partial e_i} = \frac{r_{ii}\theta_i^{(i)}}{e_i} = \frac{-r_{ij}\theta_i^{(j)}}{e_i}, \quad \theta_i^{(i)} > 0, \quad \theta_i^{(j)} < 0, \quad r_{ii}\theta_i^{(i)} = -r_{ij}\theta_i^{(j)}.$$

2. $1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}) > \left[\delta_{ii} r_{ii} + \delta_{ij}\right]\theta_i^{(i)}$*, where the inequality holds strictly in the limit when* $r_{ii} \to 0$.
3. $1 - l_{ii} - \delta_{ii}\theta_i\left(r_{ij}^{-1}\right) > -\left[\delta_{ii} + \delta_{ij} r_{ij}\right] r_{ij}\theta_i^{(j)}$.
4. $r_{ii}\phi_{ij}(r_{ii}) = r_{ij}\phi_{ii}(r_{ij})$.
5. $1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}) > 0$*, and*

$$\lim_{r_{ii} \to 0+} (1 - l_{ii} - \delta_{ii}\theta_i(r_{ii})) \in (0, +\infty), \qquad \lim_{r_{ij} \to 0+} \left(1 - l_{ii} - \delta_{ii}\theta_i\left(r_{ij}^{-1}\right)\right) \in (0, +\infty).$$

6. $\frac{\partial \theta_i^{(i)}}{\partial r_{ii}} \leq 0$*, and* $\lim_{r_{ii} \to 0+} \theta_i^{(i)} \in (0, +\infty)$.
7. $\frac{\partial \theta_i^{(j)}}{\partial r_{ij}} \geq 0$*, and* $\lim_{r_{ij} \to 0+} \theta_i^{(j)} \in (-\infty, 0)$.
8. $\phi_{ij}(r_{ii}) > 0$*, and* $\lim_{r_{ii} \to 0+} \phi_{ij}(r_{ii}) \in (0, +\infty)$.
9. $\phi_{ii}(r_{ij}) > 0$*, and* $\lim_{r_{ij} \to 0+} \phi_{ii}(r_{ij}) \in (0, +\infty)$.

Proof of Lemma 4

Part 1

The chain rule and some algebra will give

$$\frac{\partial r_{ii}}{\partial e_i} = \frac{\partial}{\partial e_i}\left(\frac{e_i}{e_j^{k_i}}\right) = \frac{1}{e_j^{k_i}} = \frac{r_{ii}}{e_i}, \qquad \frac{\partial r_{ij}}{\partial e_i} = \frac{\partial}{\partial e_i}\left(\frac{e_j^{k_i}}{e_i}\right) = -\frac{e_j^{k_i}}{e_i^2} = -\frac{r_{ij}}{e_i},$$

and, using equation (5),

$$\frac{\partial \theta_i}{\partial e_i} = \frac{\partial \theta_i}{\partial r_{ii}}\frac{\partial r_{ii}}{\partial e_i} = \frac{\partial \theta_i}{\partial r_{ii}}\frac{r_{ii}}{e_i}, \qquad \frac{\partial \theta_i}{\partial e_i} = \frac{\partial \theta_i}{\partial r_{ij}}\frac{\partial r_{ij}}{\partial e_i} = -\frac{\partial \theta_i}{\partial r_{ij}}\frac{r_{ij}}{e_i}.$$

Then an application of the chain rule using these results and the property $\frac{\partial \theta_i}{\partial e_i} > 0$ from Assumption 2 gives $\theta_i^{(i)} > 0$ and $\theta_i^{(j)} < 0$.

Parts 2-3

Some algebra using Assumption 3 and part 1 gives the result.

Part 4

Some algebra using part 1 gives the result.

Part 5

Some algebra using Assumption 5 and part 1 gives the result.

Part 6

Using part 1 and equation (5), some algebra reveals

$$\theta_i(e_i, e_j) = e_j^{-2k_i} \frac{\partial \theta_i^{(i)}}{\partial r_{ii}} \leq 0$$

where the last inequality comes from Assumption 2; this implies $\frac{\partial \theta_i^{(i)}}{\partial r_{ii}} \leq 0$. This result, the property $\theta_i^{(i)} = e_j^{k_i} \frac{\partial \theta_i}{\partial e_i} > 0$ (from part 1) and the upper-bound aspect of Assumption 2 implies $\lim_{r_{ii} \to 0+} \theta_i^{(i)} \in (0, +\infty)$.

Part 7

Conceive a function $\hat{e}_j(r_{ij}) = r_{ij}^{1/k_i}$ and use equations (5), (27) to obtain

$$\theta_i^{(j)} = \frac{\partial}{\partial r_{ij}} \theta_i\left(1, r_{ij}^{1/k_i}\right) = \left[\frac{\partial}{\partial r_{ij}}\left(\hat{e}_j^{k_i}\right)\right]\left[\frac{\partial}{\partial \hat{e}_j^{k_i}} \theta_i\left(1, \hat{e}_j\right)\right] = \frac{\partial}{\partial \hat{e}_j^{k_i}} \theta_i\left(1, \hat{e}_j\right)$$

where the second last equality follows from the chain rule. Then

$$\lim_{r_{ij} \to 0+} \theta_i^{(j)} = \lim_{\hat{e}_j^{k_i} \to 0+} \left(\frac{\partial}{\partial \hat{e}_j^{k_i}} \theta_i\left(1, \hat{e}_j\right)\right)$$

$$\frac{\partial \theta_i^{(j)}}{\partial r_{ij}} = \left(\frac{\partial}{\partial r_{ij}}\left(\hat{e}_j^{k_i}\right)\right)\left(\frac{\partial^2}{\partial\left(\hat{e}_j^{k_i}\right)^2} \theta_i\left(1, \hat{e}_j\right)\right) = \frac{\partial^2}{\partial\left(\hat{e}_j^{k_i}\right)^2} \theta_i\left(1, \hat{e}_j\right).$$

The results then follow from Assumption 2 and the property $\theta_i^{(j)} < 0$ in part 1.

Part 8

Parts 2 and 6 respectively prove that the denominator and numerator of $\phi_{ij}(r_{ii})$ (defined by equation (7)) are positive; thus $\phi_{ij}(r_{ii}) > 0$. Then some algebra using part 6 and the properties of limits reveals

$$\lim_{r_{ii} \to 0+} \phi_{ij}(r_{ii}) = \frac{v_i\left(\lim_{r_{ii} \to 0+} \theta_i^{(i)}(r_{ii})\right)}{\lim_{r_{ii} \to 0+}\left(1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}) - \delta_{ij}\theta_i^{(i)}(r_{ii})\right)}$$

where parts 5 and 2 respectively prove the numerator and denominator of the right-hand side are (strictly) positive and bounded above, giving the result.

Part 9

Parts 3 and 7 respectively prove that the denominator and numerator of $\phi_{ii}(r_{ij})$ (defined by equation (28)) are positive; thus $\phi_{ii}(r_{ij}) > 0$. Then some algebra using part 6 and the properties of limits reveals

$$\lim_{r_{ij}\to 0+} \phi_{ii}(r_{ij}) = \frac{-v_i \lim_{r_{ij}\to 0+} \theta_i^{(j)}(r_{ij})}{\lim_{r_{ij}\to 0+}\left(1 - l_{ii} - \delta_{ii}\theta_i\left(r_{ij}^{-1}\right)\right)}$$

where parts 5 and 7 respectively prove the denominator and numerator of the right-hand side are positive and bounded above, giving the result. □

Proof of Lemma 1

Take the partial derivatives of player i's payoff function in equation (2) with respect to her expenses e_i to obtain

$$\frac{\partial U_i}{\partial e_i} = \frac{\partial \theta_i}{\partial e_i}\left[v_i + \delta_{ii}e_i + \delta_{ij}e_j^{k_i}\right] + \delta_{ii}\theta_i(e_i, e_j) - (1 - l_{ii}) \tag{29}$$

$$\frac{\partial^2 U_i}{\partial e_i^2} = \frac{\partial^2 \theta_i}{\partial e_i^2}\left[v_i + \delta_{ii}e_i + \delta_{ij}e_j^{k_i}\right] + 2\delta_{ii}\frac{\partial \theta_i}{\partial e_i}. \tag{30}$$

Supposing player i's FOC holds and using equation (29), (30), a substitution exercise reveals

$$\begin{aligned}\frac{\partial^2 U_i}{\partial e_i^2} &= \frac{\partial^2 \theta_i}{\partial e_i^2}\left[\frac{\left(1 - l_{ii} - \delta_{ii}\theta_i(e_i, e_j)\right)}{\partial \theta_i / \partial e_i}\right] + 2\delta_{ii}\frac{\partial \theta_i}{\partial e_i} \\ &= \left(1 - l_{ii} - \delta_{ii}\theta_i(e_i, e_j)\right)\left[\frac{\left(1 - l_{ii} - \delta_{ii}\theta_i(e_i, e_j)\right)\frac{\partial^2 \theta_i}{\partial e_i^2} + 2\delta_{ii}\left(\frac{\partial \theta_i}{\partial e_i}\right)^2}{\left(1 - l_{ii} - \delta_{ii}\theta_i(e_i, e_j)\right)\frac{\partial \theta_i}{\partial e_i}}\right] < 0\end{aligned} \tag{31}$$

where the last inequality follows from some algebra using Assumption 4 and part 5 of Lemma 4.

Corollary 9.3 of Diewert et al. (1981) holds that a twice continuously differentiable function f defined on an open S is strictly quasiconcave if and only if (i) $x^0 \in S$, $v^T v = 1$ and $v^T \nabla f(x^0) v = 0$ implies $v^T \nabla^2 f(x^0) v < 0$; or ($ii$) $v^T \nabla^2 f(x^0) v = 0$ and $g(t) \equiv f(x^0 + tv)$ does not attain a local minimum at $t = 0$. Fixing player j's expenses e_j, inequality (31) implies player i's payoff function U_i is strictly quasiconcave in her expenses e_i. □

Proof of Lemma 2

Fix an arbitrary $e_j' > 0$, and let $U_i(\cdot, e_j')$ denote the player i's payoff function U_i restricted to one variable, e_i. Suppose there exists some $e_i' > 0$ that satisfies the FOC for $U_i(\cdot, e_j')$. Then the proof of Lemma 1 (see inequality inequality (31)) proves that e_i' also satisfies the SOC for $U_i(\cdot, e_j')$. Hence e_i' is a local maximum of $U_i(\cdot, e_j')$. Then Lemma 1 implies e_i' is a global maximum of $U_i(\cdot, e_j')$. □

Proof of Lemma 3

This proof will consider three different cases: (*i*) $k_1 k_2 = 1$; (*ii*) $k_1 k_2 < 1$; (*iii*) $1 < k_1 k_2 \leq 2$ (Assumption 6 imposes the upper bound $2 \geq k_1 k_2$).

Case (i): $k_1 k_2 = 1$

Use equations (7) and (28) to define functions $F, \bar{F} : \mathbb{R}_{++} \rightarrow \mathbb{R}$ by

$$F(r_2) = r_2 \phi_{11}(r_2) - \left[\phi_{21}\left(r_2^{1/k_1}\right)\right]^{1/k_2}, \tag{32}$$

$$\bar{F}(r_1) = \phi_{12}(r_1) - r_1 \left[\phi_{22}\left(r_1^{1/k_1}\right)\right]^{1/k_2},$$

where $r_2, r_1 > 0$ are positive real numbers. Some algebra using parts 1,4 of Lemma 4 and the specification $r_1 = 1/r_2$ reveals

$$r_2 F(r_2) = \bar{F}(r_1) \tag{33}$$

Some algebra using parts 8-9 in Lemma 4 and the properties of limits obtains

$$\lim_{r_2 \rightarrow 0+} F(r_2) = -\lim_{r_2 \rightarrow 0+} \left[\phi_{21}\left(r_2^{1/k_1}\right)\right]^{1/k_2} < 0,$$

$$\lim_{r_1 \rightarrow 0+} \bar{F}(r_1) = \lim_{r_1 \rightarrow 0+} \phi_{12}(r_1) > 0.$$

Hence there exist sufficiently small positive real numbers $\epsilon_1, \epsilon_2 > 0$ satisfying

$$0 > F(\epsilon_2), \tag{34}$$

$$0 < \bar{F}(\epsilon_1) = \epsilon_1^{-1} F\left(\epsilon_1^{-1}\right), \tag{35}$$

where the last equality uses equation (33).

Conditions (34), (35) and the intermediate value theorem imply there exists some positive real number $r_2^* \in \left(\epsilon_2, \epsilon_1^{-1}\right)$ such that $F(r_2^*) = 0$. Choosing $r_{11}^* = 1/r_2^*$, $r_{22}^* = r_2^{*k_1}$ gives the result in part 1 of Lemma 3.

Case (ii): $k_1 k_2 < 1$

Define functions $R_{22}, R_{21}, G, \bar{G} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ by

$$R_{22}(r_{12}) = r_{12}^{(2-k_1k_2)/k_1}[\phi_{11}(r_{12})]^{(1-k_1k_2)/k_1}, \tag{36}$$

$$R_{21}(r_{11}) = r_{11}^{k_2}[\phi_{12}(r_{11})]^{(k_1k_2-1)/k_1,} \tag{37}$$

$$G(r_{12}) = r_{12}^{(k_1k_2-1)k_2}[\phi_{11}(r_{12})]^{(k_1k_2-1)k_2} - [\phi_{21}(R_{22}(r_{12}))]^{k_1k_2-1}, \tag{38}$$

$$\bar{G}(r_{11}) = r_{11}^{(1-k_1k_2)k_2}[\phi_{12}(r_{11})]^{(k_1k_2-1)(2-k_1k_2)/k_1} - [\phi_{22}(R_{21}(r_{11}))]^{k_1k_2-1}, \tag{39}$$

where $r_{12}, r_{11} > 0$ are positive real numbers and part 9 of Lemma 4 imply

$$R_{21}(r_{11}) > 0. \tag{40}$$

Some algebra using parts 1,4 of Lemma 4 and the specification $r_{12} = 1/r_{11}$ reveals

$$R_{21}(r_{11}) = [R_{22}(r_{21})]^{-1}, \tag{41}$$

$$\frac{r_{12}^{k_2}}{[R_{22}(r_{12})]^{k_1k_2}} = r_{12}^{(k_1k_2-1)k_2}[\phi_{11}(r_{12})]^{(k_1k_2-1)k_2}, \tag{42}$$

$$\frac{[R_{21}(r_{11})]^{2-k_1k_2}}{r_{11}^{k_2}} = r_{11}^{(1-k_1k_2)k_2}[\phi_{12}(r_{11})]^{(k_1k_2-1)(2-k_1k_2)/k_1}, \tag{43}$$

$$G(r_{12}) = [R_{21}(r_{11})]^{2(k_1k_2-1)}\bar{G}(r_{11}), \tag{44}$$

where the last equality follows from some algebra commencing with substituting equations (43), (44) into equations (38), (39) respectively.

Some algebra using parts 8-9 of Lemma 4, the specification $k_1k_2 \leq 2$ and the properties of limits obtains

$$\lim_{r_{12}\to 0+} R_{22}(r_{12}) \in [0, +\infty), \tag{45}$$

$$\lim_{r_{11}\to 0+} R_{21}(r_{11}) = 0. \tag{46}$$

Using these limit properties, those in parts 8-9 of Lemma 4, and the specification $k_1k_2 < 1$, some algebra obtains

$$\lim_{r_{12}\to 0+} G(r_{12}) = +\infty, \qquad \lim_{r_{11}\to 0+} \bar{G}(r_{11}) < 0,$$

which respectively imply the existence of some small positive real numbers $\alpha_1, \alpha_2 > 0$ such that

$$0 < G(\alpha_2), \tag{47}$$

$$0 > \bar{G}(\alpha_1) = [R_{21}(\alpha_1)]^{2(k_1k_2-1)} G\left(\alpha_1^{-1}\right), \tag{48}$$

where the last equality follows from equation (44).

Conditions (40), (47), (48) and the intermediate value theorem imply there exists some positive real number $r_{12}^* \in \left(\alpha_1^{-1}, \alpha_2\right)$ such that $G\left(r_{12}^*\right) = 0$. Choosing $r_{11}^* = 1/r_{12}^*$ and $r_{22}^* = R_{22}\left(r_{12}^*\right)$ as defined by (36) and using equations (38) and (42), some algebra gives the result in part 2 of Lemma 3 when $k_1k_2 < 1$.

Case (iii): $1 < k_1k_2 \leq 2$

Consider functions $R_{22}(r_{12})$, $R_{21}(r_{11})$, $G(r_{12})$ and $\bar{G}(r_{11})$ respectively defined by equations (36)-(39). These functions continue to satisfy conditions (40)-(48) under the specification $1 < k_1k_2 \leq 2$. Using conditions (47)-(48), parts 8-9 of Lemma 4, and the specification $1 < k_1k_2 \leq 2$, some algebra obtains

$$\lim_{r_{12}\to 0+} G(r_{12}) < 0, \qquad \lim_{r_{11}\to 0+} \bar{G}(r_{11}) = +\infty,$$

which respectively imply the existence of some small positive real numbers $\beta_1, \beta_2 > 0$ such that

$$0 > G(\beta_2), \tag{49}$$

$$0 < \bar{G}(\beta_1) = [R_{21}(\beta_1)]^{2(k_1k_2-1)} G\left(\beta_1^{-1}\right), \tag{50}$$

where the last equality follows from equation (44).

Conditions (40), (49), (50) and the intermediate value theorem imply there exists some positive real number $r_{12}^* \in \left(\beta_2, \beta_1^{-1}\right)$ such that $G\left(r_{12}^*\right) = 0$. Choosing $r_{11}^* = 1/r_{12}^*$ and $r_{22}^* = R_{22}\left(r_{12}^*\right)$, some algebra gives the result in part 2 of Lemma 3 when $1 < k_1k_2 \leq 2$. □

Proof of Proposition 1

This proof will establish that the characterizations of e_1^*, e_2^* in this Proposition simultaneously satisfy the players' FOCs (system (4)) in two different cases: (*i*) $k_1k_2 = 1$; (*ii*) $k_1k_2 \neq 1$.

Case (i): $k_1k_2 = 1$

Assuming $k_1 k_2 = 1$, some algebra reveals that, for each $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, part 1 of Lemma 3 is equivalent to

$$r_{ii}^{*} r_{jj}^{*k_i} = 1 \tag{51}$$

$$r_{ii}^{*} \phi_{ij}(r_{ii}^{*}) = \left[\phi_{ji}(r_{jj}^{*})\right]^{k_i}. \tag{52}$$

To establish FOC satisfaction for player i, use the expression for e_j^* and equations (51)-(52) to obtain

$$e_j^{*k_i} = r_{jj}^{*k_i}\left[\phi_{ji}(r_{jj}^{*})\right]^{*k_i} = r_{jj}^{*k_i}\left[r_{ii}^{*}\phi_{ij}(r_{ii}^{*})\right]^{*k_i k_j} = r_{jj}^{*k_i} r_{ii}^{*k_i k_j}\left[\phi_{ij}(r_{ii}^{*})\right]^{*k_i k_j} = \phi_{ij}(r_{ii}^{*})$$

$$e_j^{*k_i} = e_i^{*}/r_{ii}^{*} \tag{53}$$

where the second last equality uses part 1 of Lemma 3 and the assumption $k_1 k_2 = 1$, and the last equality uses the expression for e_i^*.

Then some algebra using equation (7) and the expression for e_i^* obtains

$$\frac{v_i \theta_i^{(i)}(r_{ii}^{*}) r_{ii}^{*}}{1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}^{*}) - \left(\delta_{ii} r_{ii}^{*} + \delta_{ij}\right)\theta_i^{(i)}(r_{ii}^{*})} = e_i^{*} \tag{54}$$

$$v_i \theta_i^{(i)}(r_{ii}^{*}) r_{ii}^{*} + \left(\delta_{ii} r_{ii}^{*} + \delta_{ij}\right)\theta_i^{(i)}(r_{ii}^{*}) e_i^{*} = [1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}^{*})] e_i^{*} \tag{55}$$

$$\theta_i^{(i)}(r_{ii}^{*}) r_{ii}^{*}\left[v_i + \left(\delta_{ii} r_{ii}^{*} + \delta_{ij}\right)\frac{e_i^{*}}{r_{ii}^{*}}\right] = [1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}^{*})] e_i^{*} \tag{56}$$

$$\frac{\theta_i^{(i)}(r_{ii}^{*}) r_{ii}^{*}}{e_i^{*}}\left[v_i + \delta_{ii} e_i^{*} + \frac{\delta_{ij} e_i^{*}}{r_{ii}^{*}}\right] = 1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}^{*}) \tag{57}$$

$$\left.\frac{\partial \theta_i}{\partial e_i}\right|_{(e_i, e_j) = (e_i^*, e_j^*)}\left[v_i + \delta_{ii} e_i^{*} + \delta_{ij} e_j^{*k_i}\right] = 1 - l_{ii} - \delta_{ii}\theta_i(e_i^{*}, e_j^{*}) \tag{58}$$

where the last equality uses equations (5), (53) and part 1 of Lemma 4, establishing player i's FOC in system (4).

Case (ii): $k_1 k_2 \neq 1$

Assuming $k_1 k_2 \neq 1$, some algebra reveals that, for each $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, part 2 of

Lemma 3 is equivalent to

$$r_{ii}^{*k_i k_j} r_{jj}^{*k_i} = \left[\phi_{ij}(r_{ii}^*)\right]^{1-k_i k_j} \tag{59}$$

$$r_{jj}^{*k_i k_j} r_{ii}^{*k_j} = \left[\phi_{ji}(r_{jj}^*)\right]^{1-k_i k_j}. \tag{60}$$

Then use the expression for e_j^* and equation (60) to obtain

$$e_j^{*k_i} = r_{jj}^{*k_i}\left[\phi_{ji}(r_{jj}^*)\right]^{*k_i} = r_{jj}^{*k_i}\left[r_{jj}^{*k_i k_j} r_{ii}^{*k_j}\right]^{k_i/(1-k_i k_j)} = \left(r_{jj}^*\right)^{k_i+[k_i^2 k_j/(1-k_i k_j)]}\left(r_{ii}^*\right)^{k_i k_j/(1-k_i k_j)}$$

$$= \left(r_{jj}^*\right)^{k_i/(1-k_i k_j)}\left(r_{ii}^*\right)^{k_i k_j/(1-k_i k_j)} = \phi_{ij}(r_{ii}^*) = e_i^*/r_{ii}^*$$

where the second last equality uses equation (59), and the last equality uses the expression for e_i^*. Then the steps establishing equations (54)-(58) prove FOC satisfaction for player $i \in \{1,2\}\setminus\{j\}$. □

Proof of Corollary 1

Suppose Lemma 3 defines a unique pair (r_{11}^*, r_{22}^*). Suppose, for a contradiction, there exists a nontrivial Nash equilibrium of the Contest Game, denoted $(\bar{e}_1, \bar{e}_2)$, such that $(\bar{e}_1, \bar{e}_2) \neq (e_1^*, e_2^*)$ given by Proposition 1.

Define auxiliary constants $\bar{r}_{11} = \bar{e}_1/\bar{e}_2^{k_1}$ and $\bar{r}_{22} = \bar{e}_2/\bar{e}_1^{k_2}$. Using the property that $(\bar{e}_1, \bar{e}_2)$ satisfies the FOC in system (4) and reversing the steps establishing equations (54)-(58), some algebra obtains

$$\bar{e}_1 = \bar{r}_{11}\phi_{12}(\bar{r}_{11}) \tag{61}$$

$$\bar{e}_2^{k_1} = \phi_{12}(\bar{r}_{11}) \tag{62}$$

$$\bar{e}_2 = \bar{r}_{22}\phi_{21}(\bar{r}_{22}) \tag{63}$$

$$\bar{e}_1^{k_2} = \phi_{21}(\bar{r}_{22}). \tag{64}$$

Then some algebra comparing equation (61) with equation (64), and equation (62) with equation (63), obtains

$$\phi_{21}(\bar{r}_{22}) = \bar{r}_{11}^{k_2}[\phi_{12}(\bar{r}_{11})]^{k_2} \tag{65}$$

$$\phi_{12}(\bar{r}_{11}) = \bar{r}_{22}^{k_1}[\phi_{21}(\bar{r}_{22})]^{k_1} \tag{66}$$

where a substitution exercise using these equations reveals

$$\bar{r}_{22}^{k_1 k_2} \bar{r}_{11}^{k_2} = [\phi_{21}(\bar{r}_{22})]^{1-k_1 k_2} \tag{67}$$

$$\bar{r}_{11}^{k_1 k_2} \bar{r}_{22}^{k_1} = [\phi_{12}(\bar{r}_{11})]^{1-k_1 k_2}. \tag{68}$$

Consider two different cases: (i) $k_1 k_2 \neq 1$; (ii) $k_1 k_2 = 1$.

Case (i): $k_1 k_2 \neq 1$

Equations (67) and (68), the assumption $k_1 k_2 \neq 1$ and the assumption that Lemma 3 defines a unique pair (r_{11}^*, r_{22}^*) imply $(\bar{r}_{11}, \bar{r}_{22}) = (r_{11}^*, r_{22}^*)$. Then an application of Proposition 1 obtains $(\bar{e}_1, \bar{e}_2) = (e_1^*, e_2^*)$, a contradiction.

Case (ii): $k_1 k_2 = 1$

The assumption $k_1 k_2 = 1$ and equation (67) imply

$$\bar{r}_{11} \bar{r}_{22}^{k_1} = 1. \tag{69}$$

A substitution exercise using equations (66) and (69) and the assumption $k_1 k_2 = 1$ obtains

$$[\bar{r}_{11} \phi_{12}(\bar{r}_{11})]^{k_2} = \phi_{21}(\bar{r}_{22}). \tag{70}$$

Equations (69) and (70), the assumption $k_1 k_2 = 1$ and the assumption that Lemma 3 defines a unique pair (r_{11}^*, r_{22}^*) imply $(\bar{r}_{11}, \bar{r}_{22}) = (r_{11}^*, r_{22}^*)$. Then an application of Proposition 1 obtains $(\bar{e}_1, \bar{e}_2) = (e_1^*, e_2^*)$, a contradiction. □

Proof of Corollary 2

Following similar steps in the proof of Corollary 6, some algebra reveals $F(r_2) = 0$ and $r_2 > 0$ if and only if

$$r_2 = \frac{\theta_1}{\theta_2} = \frac{\eta \mu_1 + \frac{1-\eta}{1+r_2}}{\eta \mu_2 + \frac{(1-\eta) r_2}{1+r_2}} \tag{71}$$

Some algebra using equation (10) reveals equation (71) is equivalent to

$$[1 - \eta(1 - \mu_2)] r_2^2 - \eta[(1 - \mu_2) - (1 - \mu_1)] - [1 - \eta(1 - \mu_1)] = 0.$$

Using the quadratic formula, some algebra obtains

$$r_2 = \frac{\eta[(1 - \mu_2) - (1 - \mu_1)] \pm \sqrt{\Delta}}{2[1 - \eta(1 - \mu_2)]} \tag{72}$$

where

$$\begin{aligned}
\Delta =&\eta^2[(1-\mu_2)-(1-\mu_1)]^2+4[1-\eta(1-\mu_2)][1-\eta(1-\mu_1)]\\
=&\eta^2[(1-\mu_2)-(1-\mu_1)]^2+4[1-\eta[(1-\mu_2)+(1-\mu_1)]]-4\eta^2(1-\mu_2)(1-\mu_1)\\
=&\eta^2[(1-\mu_2)^2-2(1-\mu_2)(1-\mu_1)+(1-\mu_1)^2-4(1-\mu_2)(1-\mu_1)]\\
&+4[1-\eta[(1-\mu_2)+(1-\mu_1)]]\\
=&\eta^2[(1-\mu_2)^2+2(1-\mu_2)(1-\mu_1)+(1-\mu_1)^2]+4[1-\eta[(1-\mu_2)+(1-\mu_1)]]\\
=&\eta^2[(1-\mu_2)+(1-\mu_1)]^2-4\eta[(1-\mu_2)+(1-\mu_1)]+4\\
=&[2-\eta[(1-\mu_2)+(1-\mu_1)]]^2
\end{aligned}$$

where $2-\eta[(1-\mu_2)+(1-\mu_1)]>0$ because $0\leq\eta,\mu_1,\mu_2,\leq 1$. Substituting back to equation (72) obtains the only positive solution

$$r_2=\frac{1-\eta+\eta\mu_1}{1-\eta+\eta\mu_2}.$$

Then a substitution exercise using Proposition 1 gives the result. □

Proof of Corollary 3

Some algebra using Corollary 2 reveals part 1 and

$$\theta_i^*-\theta_j^*=\frac{(\mu_i-\mu_j)\eta[\eta(\mu_1+\mu_2)+1-\eta]}{2(1-\eta)+\eta(\mu_1+\mu_2)}$$

where the right-hand side is negative if and only if $\mu_i<\mu_j$, giving part 2. □

Proof of Corollary 4

Some algebra using Corollary 2 obtains

$$e_1^*+e_2^*=\frac{v(1-\eta)}{\lambda[2(1-\eta)+\eta(\mu_1+\mu_2)]}=\frac{v(1-\eta)}{\lambda[2-\eta+\eta\sigma]}$$

then a differentiation exercise gives the result.

Proof of Corollary 5

Some algebra using the product rule and the property $\theta_1^*+\theta_2^*=1+\eta\sigma$ from equation (11) reveals

$$\frac{dU^*}{d\sigma} = \left[v - \lambda\left(e_1^* + e_2^*\right)\right]\frac{d}{d\sigma}(1+\eta\sigma) - [1+\eta\sigma]\lambda\frac{d}{d\sigma}\left(e_1^* + e_2^*\right)$$
$$= \left[v - \lambda\left(e_1^* + e_2^*\right)\right]\eta - (1+\eta\sigma)\lambda\frac{d}{d\sigma}\left(e_1^* + e_2^*\right)$$

where Corollary 4 proves $\frac{d}{d\sigma}\left(e_1^* + e_2^*\right) < 0$ and some algebra using the properties $0 \leq \mu_1, \mu_2 \leq 1$ and Corollary 2 obtains

$$1 + \eta\sigma > 0, \qquad v - \lambda\left(e_1^* + e_2^*\right) = \frac{v(1+\eta\sigma)}{2 - \eta + \eta\sigma} > 0,$$

giving the result. □

Proof of Corollary 6

Denote an auxiliary variable $r_2 = r_{12}$. The assumption $k_1 = k_2$ implies $r_{22} = r_{12} = r_2$. Some algebra reveals that the FOCs in system (4) hold simultaneously if and only if the function $F(r_2) = 0$, where equation (32) defines F. Some algebra further reveals $F(r_2) = 0$ and $r_2 > 0$ if and only if $r_2 = 1$. Then a substitution exercise using Proposition 1 gives the result. □

Proof of Corollary 7

Using Corollary 6, some algebra reveals

$$\frac{d}{d\delta}\left(e_1^* + e_2^*\right) = \frac{2v\mu^2(1-\mu)^2}{[1 - \delta\mu(1-\mu)]^2}$$

$$\frac{d}{d(\mu(1-\mu))}\left(\frac{d}{d\delta}\left(e_1^* + e_2^*\right)\right) = \frac{4v\mu(1-\mu)}{[1 - \delta\mu(1-\mu)]^3}. \tag{73}$$

The result follows from inequality (73) and the assumption $\delta < \min\{\mu^{-1}(1-\mu), (1-\mu)^{-1}\mu\}$ (which implies $1 - \delta\mu(1-\mu) > 0$). □

Proof of Corollary 8

Some algebra using Corollary 6 obtains

$$U_1^* + U_2^* = \frac{v[1 - 2\mu(1-\mu)(1-l)]}{1 - (w-l)\mu(1-\mu)}.$$

Then a calculus exercise reveals

$$\frac{d}{dw}\left(U_1^* + U_2^*\right) = \frac{v\mu(1-\mu)}{[1 - (w-l)\mu(1-\mu)]^2}$$

$$\frac{d}{dl}\left(U_1^* + U_2^*\right) = \frac{v\mu(1-\mu)[1+2\mu(1-\mu)(1-w)]}{[1-(w-l)\mu(1-\mu)]^2}$$

$$\frac{d}{d\tau}\left(U_1^* + U_2^*\right) = \frac{v(w+l-2)}{[1-(w-l)\mu(1-\mu)]^2}$$

giving the results. □

Proof of Corollary 9

Using Corollary 6, a calculus exercise reveals

$$\frac{d}{d\tau}\left(e_1^* + e_2^*\right) = \frac{2v}{[1-\delta\tau]^2} > 0$$

where the last inequality follows from the assumption $\delta < \min\{\mu^{-1}(1-\mu), (1-\mu)^{-1}\mu\}$ and the property $\tau = \mu(1-\mu) \leq 0.25$. □

Proof of Corollary 10

Steps similar to those in the proof of Corollary 6 give the result. □

Proof of Corollary 11

Consider Corollary 10, which finds the close-form expression for the nontrivial Nash equilibrium of the Alternative R&D Game. Part 1 of Corollary 11 follows from the absence of τ — the variable that measures the relative advantages of the players — in the close-form expression. Part 2 of Corollary 11 follows from a calculus exercise using the close-form expression. □

Proof of Proposition 2

This proof applies the principle of finite induction. The result clearly holds for the base case $\theta_i = \theta_{i(1)}$. The rest of this proof assumes the result holds for all integer $z \leq n-1$, and proves that it holds for $z = n$. Define auxiliary constants $0 \leq a, \bar{a} \leq 1$ and a success function $\widetilde{\theta} : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$a = p_{i(1)} + p_{i(2)} + \ldots + p_{i(n-1)}, \qquad \bar{a} = 1 - a, \qquad \widetilde{\theta} = \sum_{z=1}^{n-1}\left(\frac{p_{i(z)}}{a}\theta_{i(z)}\right).$$

Assuming $\widetilde{\theta}, \theta_{i(n)}$ satisfy Assumption 1 (respectively, 2, 5, 6), some algebra immediately reveals that their linear combination $\theta_i = a\widetilde{\theta} + \bar{a}\theta_{i(n)}$ also satisfies Assumption 1 (2, 5, 6). The following will prove that θ_i satisfies the remaining Assumptions 3 and 4.

Assumption 3

Some algebra using the linearity of derivation obtains

$$\frac{\partial\theta_i}{\partial e_i}\left[\delta_{ii}e_i + \delta_{ij}e_j^{k_i}\right] + \delta_{ii}\theta_i = \frac{\partial}{\partial e_i}\left(a\widetilde{\theta} + \bar{a}\theta_{i(n)}\right)\left[\delta_{ii}e_i + \delta_{ij}e_j^{k_i}\right] + \delta_{ii}\left(a\widetilde{\theta} + \bar{a}\theta_{i(n)}\right)$$

$$= a\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\left[\delta_{ii}e_i + \delta_{ij}e_j^{k_i} + \delta_{ii}\widetilde{\theta}\right]\right) + \bar{a}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\left[\delta_{ii}e_i + \delta_{ij}e_j^{k_i}\right] + \delta_{ii}\theta_{i(n)}\right)$$

$$< a(1 - l_{ii}) + \bar{a}(1 - l_{ii}) = 1 - l_{ii}$$

where the inequality uses the assumption that $\widetilde{\theta}$, $\theta_{i(n)}$ satisfy Assumptions 3. Hence θ_i satisfies Assumption 3.

Assumption 4

Assuming $\widetilde{\theta}, \theta_n$ satisfy Assumption (4), some algebra reveals

$$a\frac{\partial^2\widetilde{\theta}}{\partial e_i^2} + \frac{2a\delta_{ii}\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2}{\left(1 - l_{ii} - \delta_{ii}\widetilde{\theta}\right)} < a\frac{\partial\widetilde{\theta}}{\partial e_i}, \qquad \bar{a}\frac{\partial^2\theta_{i(n)}}{\partial e_i^2} + \frac{2\bar{a}\delta_{ii}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2}{\left(1 - l_{ii} - \delta_{ii}\theta_{i(n)}\right)} < \bar{a}\frac{\partial\theta_{i(n)}}{\partial e_i}.$$

Some algebra summing these inequalities and using the definition of θ_i and the linearity of differentiation obtains

$$\frac{\partial^2\theta_i}{\partial e_i^2} + 2\delta_{ii}\left[\frac{a\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2}{\left(1 - l_{ii} - \delta_{ii}\widetilde{\theta}\right)} + \frac{\bar{a}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2}{\left(1 - l_{ii} - \delta_{ii}\theta_{i(n)}\right)}\right] < \frac{\partial\theta_i}{\partial e_i}. \tag{74}$$

Some algebra using the non-negative property of squares obtains

$$a\bar{a}\left[\frac{\partial\widetilde{\theta}}{\partial e_i}\left(1 - l_{ii} - \delta_{ii}\theta_{i(n)}\right) - \frac{\partial\theta_{i(n)}}{\partial e_i}\left(1 - l_{ii} - \delta_{ii}\widetilde{\theta}\right)\right]^2 \geq 0$$

$$2a\bar{a}\frac{\partial\widetilde{\theta}}{\partial e_i}\frac{\partial\theta_{i(n)}}{\partial e_i}\left(1 - l_{ii} - \delta_{ii}\widetilde{\theta}\right)\left(1 - l_{ii} - \delta_{ii}\theta_{i(n)}\right)$$

$$\leq a\bar{a}\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2\left(1 - l_{ii} - \delta_{ii}\theta_{i(n)}\right)^2 + a\bar{a}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2\left(1 - l_{ii} - \delta_{ii}\widetilde{\theta}\right)^2$$

$$2a\bar{a}\frac{\partial\widetilde{\theta}_i}{\partial e_i}\frac{\partial\theta_{i(n)}}{\partial e_i} + a^2\left(\frac{\partial\widetilde{\theta}_i}{\partial e_i}\right)^2 + \bar{a}^2\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2$$

$$\leq \frac{a\bar{a}\left(\frac{\partial\widetilde{\theta_i}}{\partial e_i}\right)^2(1-l_{ii}-\delta_{ii}\theta_{i(n)})^2 + a\bar{a}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2\left(1-l_{ii}-\delta_{ii}\widetilde{\theta_i}\right)^2}{\left(1-l_{ii}-\delta_{ii}\widetilde{\theta_i}\right)(1-l_{ii}-\delta_{ii}\theta_{i(n)})} + a^2\left(\frac{\partial\widetilde{\theta_i}}{\partial e_i}\right)^2 + \bar{a}^2\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2$$

$$\frac{a^2\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2 + 2a\bar{a}\frac{\partial\widetilde{\theta}}{\partial e_i}\frac{\partial\theta_{i(n)}}{\partial e_i} + \bar{a}^2\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2}{a\left(1-l_{ii}-\delta_{ii}\widetilde{\theta}\right)+\bar{a}(1-l_{ii}-\delta_{ii}\theta_{i(n)})} \leq \frac{a(1-l_{ii}-\delta_{ii}\theta_{i(n)})\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2 + \bar{a}\left(1-l_{ii}-\delta_{ii}\widetilde{\theta}\right)\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2}{\left(1-l_{ii}-\delta_{ii}\widetilde{\theta}\right)(1-l_{ii}-\delta_{ii}\theta_{i(n)})}$$

$$\frac{\left[a\frac{\partial\widetilde{\theta}}{\partial e_i}+\bar{a}\frac{\partial\theta_{i(n)}}{\partial e_i}\right]^2}{1-l_{ii}-\delta_{ii}(a\widetilde{\theta}+\bar{a}\theta_{i(n)})} \leq \frac{a\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2}{\left(1-l_{ii}-\delta_{ii}\widetilde{\theta}\right)} + \frac{\bar{a}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2}{(1-l_{ii}-\delta_{ii}\theta_{i(n)})}$$

$$\frac{\left(\frac{\partial\theta_i}{\partial e_i}\right)^2}{1-l_{ii}-\delta_{ii}\theta_i} \leq \frac{a\left(\frac{\partial\widetilde{\theta}}{\partial e_i}\right)^2}{\left(1-l_{ii}-\delta_{ii}\widetilde{\theta}\right)} + \frac{\bar{a}\left(\frac{\partial\theta_{i(n)}}{\partial e_i}\right)^2}{(1-l_{ii}-\delta_{ii}\theta_{i(n)})} \tag{75}$$

where the last step uses the definition of θ_i and the linearity of differentiation. Then some algebra using inequalities (74), (75) obtains

$$\frac{(1-l_{ii}-\delta_{ii}\theta_i)\frac{\partial^2\theta_i}{\partial e_i^2}+2\delta_{ii}\left(\frac{\partial\theta_i}{\partial e_i}\right)^2}{(1-l_{ii}-\delta_{ii}\theta_i)\frac{\partial\theta_i}{\partial e_i}} < 0,$$

where some further algebra will reveal that it implies θ_i satisfies Assumption 4. □

Proof of Proposition 3

The FOC of player i in the Homogeneous Contest Game is

$$\frac{\partial\widehat{U}_i}{\partial y_i} = \frac{\partial\theta_i}{\partial y_i}\left[v_i+\Delta_{ii}(1)y_i^{k_{ii}}+\Delta_{ij}(1)y_j^{k_{ij}}\right] - [E_i(1)-L_{ii}(1)-\Delta_{ii}(1)\theta_i]k_{ii}y_i^{k_{ii}-1} = 0,$$

$$0 = E_i'(y_i)\left(\frac{\partial\theta_i}{\partial E_i}\left[v_i+\frac{\Delta_{ii}(1)}{E_i(1)}E_i(y_i)+\frac{\Delta_{ij}(1)}{E_j(1)^{\frac{k_{ij}}{k_{jj}}}}\left[E_j(y_j)\right]^{\frac{k_{ij}}{k_{jj}}}\right]-\left[1-\frac{L_{ii}(1)}{E_i(1)}-\frac{\Delta_{ii}(1)}{E_i(1)}\theta_i\right]\right), \tag{76}$$

where the last step applies the chain rule and uses the properties of homogeneous functions.

Now, let the spillover parameters in the original Contest Game be those defined by (25). Then a comparison of (76) and system (4) reveals that a pair of positive efforts (y_1^*, y_2^*) satisfies (76) if and only if it relates to a nontrivial Nash equilibrium (e_1^*, e_2^*) of the Contest Game (see Proposition 1) via the transformation in (26).

To see that the pair (y_1^*, y_2^*) satisfies player i's SOC, some algebra using the chain rule and equation (76) obtains

$$\frac{\partial^2 \widehat{U}_i}{\partial y_i^2} = E_i''(y_i)\left(\frac{\partial \widehat{U}_i}{\partial E_i}\right) + \left(E_i'(y_i)\right)^2\left(\frac{\partial^2 \widehat{U}_i}{\partial E_i^2}\right)$$

which is negative when evaluated at (y_1^*, y_2^*), because $E_i''(y_i^*) \geq 0$, $E_i'(y_i^*) > 0$ and

$$\left.\frac{\partial \widehat{U}_i}{\partial E_i}\right|_{y_i=y_i^*} = \left.\frac{\partial U_i}{\partial e_i}\right|_{e_i=e_i^*} = 0, \qquad \left.\frac{\partial^2 \widehat{U}_i}{\partial E_i^2}\right|_{y_i=y_i^*} = \left.\frac{\partial U_i^2}{\partial e_i^2}\right|_{e_i=e_i^*} < 0$$

where U_i is player i's payoff in the original Contest Game, and the last inequality established in the proof of Lemma 1.

Hence, in the Homogeneous Contest Game, player i by exerting positive effort y_i^* indeed maximizes her payoff $\widehat{U}_i$ when her opponent j exerts positive effort y_j^*. The pair (y_1^*, y_2^*) thus characterizes a nontrivial Nash equilibrium of the Homogeneous Contest Game. □

References

Akerlof, R. J. and R. T. Holden (2012). The nature of tournaments. *Economic Theory 51*(2), 289–313.

Balart, P., S. M. Chowdhury, and O. Troumpounis (2017). Linking individual and collective contests through noise level and sharing rules. *Economics Letters 155*, 126–130.

Baye, M. R. and H. C. Hoppe (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior 44*(2), 217–226.

Baye, M. R., D. Kovenock, and C. G. De Vries (1993). Rigging the lobbying process: An application of the all-pay auction. *The American Economic Review 83*(1), 289.

Baye, M. R., D. Kovenock, and C. G. De Vries (1996). The all-pay auction with complete information. *Economic Theory 8*(2), 291–305.

Baye, M. R., D. Kovenock, and C. G. De Vries (2012). Contests with rank-order spillovers. *Economic Theory 51*(2), 315–350.

Baye, M. R., D. Kovenock, and C. G. Vries (2005). Comparative analysis of litigation systems: An auction-theoretic approach. *The Economic Journal 115*, 583–601.

Beviá, C. and L. C. Corchón (2015). Relative difference contest success function. *Theory and Decision 78*(3), 377–398.

Blavatskyy, P. R. (2010). Contest success function with the possibility of a draw: axiomatization. *Journal of Mathematical Economics 46*(2), 267–276.

Bos, O. and M. Ranger (2014). All-pay auctions with polynomial rewards. *Annals of Economics and Statistics/Annales D'Économie Et De Statistique* (115/116), 361–377.

Carbonara, E., F. Parisi, and G. von Wangenheim (2015). Rent-seeking and litigation: The hidden virtues of limited fee shifting. *Review of Law and Economics 11*(2), 113–148.

Che, Y. K. and I. Gale (1998). Caps on political lobbying. *The American Economic Review 88*(3), 643–651.

Che, Y. K. and I. Gale (2000). Difference-form contests and the robustness of all-pay auctions. *Games and Economic Behavior 30*(1), 22–43.

Che, Y. K. and I. Gale (2003). Optimal design of research contests. *The American Economic Review 93*(3), 646–671.

Chen, B. and J. A. Rodrigues-Neto (2017). Cost shifting in civil litigation: A general theory. *ANU Working Papers in Economics and Econometrics 2017-651*.

Chowdhury, S. M. (2017). The all-pay auction with nonmonotonic payoff. *Southern Economic Journal 84*(2), 375–390.

Chowdhury, S. M. and O. Gürtler (2015). Sabotage in contests: A survey. *Public Choice 164*(1-2), 135–155.

Chowdhury, S. M., J. Y. Jeon, and A. Ramalingam (2018). Property rights and loss aversion in contests. *Economic Inquiry 56*(3), 1492–1511.

Chowdhury, S. M. and R. M. Sheremeta (2011a). A generalized Tullock contest. *Public Choice 147*(3/4), 413–420.

Chowdhury, S. M. and R. M. Sheremeta (2011b). Multiple equilibria in tullock contests. *Economics Letters 112*(2), 216–219.

Chowdhury, S. M. and R. M. Sheremeta (2015). Strategically equivalent contests. *Theory and Decision 78*(4), 587–601.

Chung, T.-Y. (1996). Rent-seeking contest when the prize increases with aggregate efforts. *Public Choice 87*(1/2), 55–66.

Clark, D. J. and C. Riis (1998). Contest success functions: an extension. *Economic Theory 11*(1), 201–204.

Corchón, L. and M. Dahm (2010). Foundations for contest success functions. *Economic Theory 43*(1), 81–98.

Corchón, L. C. (2007). The theory of contests: A survey. *Review of Economic Design 11*(2), 69–100.

Corchón, L. C. and A. Yıldızparlak (2013). Give peace a chance: The effect of ownership and asymmetric information on peace. *Journal of Economic Behavior and Organization 92*, 116–126.

Cornes, R. and R. Hartley (2005). Asymmetric contests with general technologies. *Economic Theory 26*(4), 923–946.

Cornes, R. and R. Hartley (2012). Risk aversion in symmetric and asymmetric contests. *Economic Theory 51*(2), 247–275.

D'Aspremont, C. and A. Jacquemin (1988). Cooperative and noncooperative R& D in duopoly with spillovers. *The American Economic Review 78*(5), 1133–1137.

Dechenaux, E., D. Kovenock, and R. M. Sheremeta (2015). A survey of experimental research on contests, all-pay auctions and tournaments. *Experimental Economics 18*(4), 609–669.

Diewert, W. E., M. Avriel, and I. Zang (1981). Nine kinds of quasiconcavity and concavity. *Journal of Economic Theory 25*(3), 397–420.

Dixit, A. (1987). Strategic behavior in contests. *The American Economic Review 77*(5), 891–898.

Einy, E., O. Haimanko, D. Moreno, A. Sela, and B. Shitovitz (2015). Equilibrium existence in Tullock contests with incomplete information. *Journal of Mathematical Economics 61*, 241–245.

Garfinkel, M. R. and S. Skaperdas (2000). Conflict without misperceptions or incomplete information: How the future matters. *The Journal of Conflict Resolution 44*(6), 793–807.

Garfinkel, M. R. and S. Skaperdas (2007). Economics of conflict: An overview. In K. Hartley and T. Sandler (Eds.), *Handbook of Defense Economics*, Volume 2, pp. 649–709. Elsevier.

Hartwick, J. M. (1984). Optimal R&D levels when firm j benefits from firm i's inventive activity. *Economics Letters 16*(1-2), 165–170.

Hillman, A. L. and J. G. Riley (1989). Politically contestable rents and transfers. *Economics and Politics 1*(1), 17–39.

Hirshleifer, J. (1989a). Conflict and rent-seeking success functions: Ratio vs. difference models of relative success. *Public Choice 63*(2), 101–112.

Hirshleifer, J. (1989b). The dimensions of power as illustrated in a steady-state model of conflict. Technical report, RAND Corp., Santa Monica C.A.

Jia, H. (2008). A stochastic derivation of the ratio form of contest success functions. *Public Choice 135*(3/4), 125–130.

Kahneman, D. and A. Tversky (1977). Intuitive prediction: Biases and corrective procedures. Technical report, Decisions and Designs Inc McLean VA.

Kamien, M. I., E. Muller, and I. Zang (1992). Research joint ventures and R&D cartels. *The American Economic Review 82*(5), 1293–1306.

Katz, A. (1987). Measuring the demand for litigation: Is the English rule really cheaper? *Journal of Law, Economics, and Organization 3*(2), 143–176.

Klemperer, P. (2003). Why every economist should learn some auction theory. Advances in Economics and Econometrics: Invited Lectures to 8th World Congress of the Econometric Society.

Konrad, K. A. (2009). *Strategy and dynamics in contests*. Oxford University Press.

Krueger, A. O. (1974). The political economy of the rent-seeking society. *The American Economic Review 64*(3), 291–303.

Lazear, E. P. and S. Rosen (1981). Rank-order tournaments as optimum labor contracts. *Journal of Political Economy 89*(5), 841–864.

Menezes, F. M. and J. Quiggin (2010). Markets for influence. *International Journal of Industrial Organization 28*(3), 307–310.

Minchuk, Y. and A. Sela (2017). Contests with insurance. Technical Report DP12456, Centre for Economic Policy Research.

Moldovanu, B. and A. Sela (2001). The optimal allocation of prizes in contests. *The American Economic Review 91*(3), 542–558.

Moldovanu, B., A. Sela, and X. Shi (2007). Contests for status. *Journal of Political Economy 115*(2), 338–363.

Nitzan, S. (1991). Collective rent dissipation. *The Economic Journal 101*(409), 1522–1534.

Olszewski, W. and R. Siegel (2016). Large contests. *Econometrica 84*(2), 835–854.

Plott, C. R. (1987). Legal fees: A comparison of the American and English rules. *Journal of Law, Economics, and Organization 3*, 185–192.

Polishchuk, L. and A. Tonis (2013). Endogenous contest success functions: A mechanism design approach. *Economic Theory 52*(1), 271–297.

Puri, M. and D. T. Robinson (2007). Optimism and economic choice. *Journal of Financial Economics 86*(1), 71–99.

Radcliffe, N. M. and W. M. Klein (2002). Dispositional, unrealistic, and comparative optimism: Differential relations with the knowledge and processing of risk information and beliefs about personal risk. *Personality and Social Psychology Bulletin 28*(6), 836–846.

Rai, B. K. and R. Sarin (2009). Generalized contest success functions. *Economic Theory 40*(1), 139–149.

Sacco, D. and A. Schmutzler (2008). All-pay auctions with negative prize externalities: Theory and experimental evidence. *University of Zurich, Socioeconomic Institute, Working Paper No. 0806. https://ideas.repec.org/p/soz/wpaper/0806.html*.

Serena, M. and L. Corchón (forthcoming 2017). Contest theory: A survey. In *Handbook of Game Theory and Industrial Organization*. Edward Elgar.

Siegel, R. (2009). All-pay contests. *Econometrica 77*(1), 71–92.

Siegel, R. (2010). Asymmetric contests with conditional investments. *The American Economic Review 100*(5), 2230–2260.

Skaperdas, S. (1996). Contest success functions. *Economic Theory 7*(2), 283–290.

Skaperdas, S. and C. Syropoulos (1997). The distribution of income in the presence of appropriative activities. *Economica 64*(253), 101–1117.

Skaperdas, S. and S. Vaidya (2012). Persuasion as a contest. *Economic Theory 51*(2), 465–486.

Tullock, G. (1967). The welfare costs of tariffs, monopolies, and theft. *Western Economic Journal 5*(3), 224–32.

Tullock, G. (1980). Efficient rent seeking. In J. Buchanan, R. Tollison, and G. Tullock (Eds.), *Towards a Theory of a Rent-Seeking Society*, pp. 97–112. Texas A&M University Press.

Vojnović, M. (2016). *Contest Theory: Incentive Mechanisms and Ranking Methods*. Cambridge University Press.

Weinstein, N. D. (1980). Unrealistic optimism about future life events. *Journal of Personality and Social Psychology 39*(5), 806.

Xiao, J. (2016). Asymmetric all-pay contests with heterogeneous prizes. *Journal of Economic Theory 163*, 178–221.

Xiao, J. (2017). All-pay contests with performance spillovers. *Mathematical Social Sciences 92*, 35–39.