Equilibrium in a Model of Production Networks*

Meng Yu[†‡] Junan Zhang[§]

November 7, 2018

Abstract. In this paper, we extend and improve the production chain model introduced by Kikuchi, Nishimura, and Stachurski (2018). Utilizing the theory of monotone concave operators, we prove the existence, uniqueness, and global stability of equilibrium price, hence improving their results on production networks with multiple upstream partners. We propose an algorithm for computing the equilibrium price function that is more than ten times faster than successive evaluations of the operator. The model is then generalized to a stochastic setting that offers richer implications for the size distribution of firms.

1 Introduction

Over the past several centuries, firms have self-organized into ever more complex production networks, spanning both state and international boundaries, and constructing and delivering a vast range of manufactured goods and services. The structures of these networks help determine the efficiency (Levine 2012; Ciccone 2002) and resilience (Carvalho 2007; Jones 2011; Bigio and La'O 2016; Acemoglu et al. 2012; Acemoglu, Ozdaglar, and Tahbaz-Salehi 2015a) of the entire economy, and also provide new insights into the directions of trade and financial policies (Baldwin and Venables 2013; Acemoglu, Ozdaglar, and Tahbaz-Salehi 2015b).

We consider a production chain model introduced by Kikuchi, Nishimura, and Stachurski (2018) that examines the formation of such structures. A single firm at the end of the production chain sells a final product to consumers. The firm can choose to produce the product by itself or subcontract a portion of it to possible multiple upstream

*The authors thank John Stachurski for his helpful comments and valuable suggestions.

†Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.

‡School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China. yumeng161@mails.ucas.ac.cn

§Research School of Economics, The Australian National University. junnan.zhang@anu.edu.au

partners, who then make similar choices until all the remaining production is completed. Here is the trade-off. Subcontracting incurs transaction costs but it is also inefficient[1] to conduct all the production in-house especially for downstream firms. A price function governs the choices firms make and is determined endogenously in equilibrium when no firm in the production chain makes positive profit.

Considering that all firms are ex ante identical, a notable feature of this model is its ability to generate a production network with multiple layers of firms different in their sizes and numbers of upstream partners. The source of the heterogeneity lies solely in the transaction costs and firms' different stages in the production chain. This feature provides insights into the formation of potentially more complex structures in a production network. Kikuchi, Nishimura, and Stachurski (2018) prove the existence, uniqueness, and global stability[2] of the equilibrium price function restricting every firm to have only one upstream partner. In this case, the resulting production network consists of a single chain.

There are however, several significant weaknesses with the analysis in Kikuchi, Nishimura, and Stachurski (2018). First, while they provide comprehensive results on uniqueness of equilibrium prices and convergence of successive approximations in the single upstream partner case, they fail to provide analogous results for the more interesting multiple upstream partner case, presumably due to technical difficulties. Second, their model cannot accurately reflect the data on observed production networks because their networks are always symmetric, with sub-networks at each layer being exact copies of one another. Real production networks do not exhibit this symmetry. Third, they provide no effective algorithm for computing the equilibrium price function in the multiple upstream partner case.

This paper resolves all of the shortcomings listed above. As our first contribution, we extend their existence, uniqueness, and global stability results to the multiple partner case. To avoid the technical difficulties faced in their paper, we employ a different approach utilizing the theory of monotone concave operators, which enables us to give a unified proof for both cases.

Theoretically, the concave operator theory ensures the global stability of the fixed point, so the equilibrium price function can be computed by successive evaluations of the operator. In practice, however, the rates of convergence can be different for different model settings. This leads to unnecessarily long computation time in most cases. As a second contribution, we propose an algorithm that achieves fast computation regardless of

1. One justification also mentioned in Kikuchi, Nishimura, and Stachurski (2018) is that firms usually experience diminishing return to management: when a firm gets bigger it also bears increasing coordination costs. See also Lucas (1978) and Becker and Murphy (1992).

2. Mathematically, the equilibrium price function is determined as the fixed point of a Bellman like operator (see Section 3). Globally stability means that the fixed point can be computed by successive evaluations of the operator on any function in a certain function space.

parameterizations and is shown to drastically reduce computation time in our simulations.

A third contribution of this paper is that we generalize the model to a stochastic setting. In the original model, the equilibrium firm allocation is symmetric and deterministic: firms at the same stage of production choose the exact same number of upstream partners. In reality, each firm faces uncertainty in the contracting process and can not always choose the optimal number of partners. We model the number of upstream partners as a Poisson distribution and let the firm choose its parameter, which can be seen as a search effort. Using the same approach, we prove the existence and uniqueness of equilibrium price function as well as the validity of the algorithm. This generalization provides a new source of heterogeneity in the equilibrium firm allocation and can be a potential channel for future research on size distribution of firms.

As briefly mentioned above, the method we use to establish the existence, uniqueness, and global stability of the equilibrium price function draws on the theory of concave operators. A competing method traditionally used for the same purpose is the Contraction Mapping Theorem, which has been an essential tool for economists dealing with various dynamic models ever since Bellman (1957). So long as the operator in question satisfies the contraction property, we can quickly compute a unique fixed point by applying the operator successively. This property, simple as it may be, is not shared among a number of important models, urging us to find new tools to tackle fixed point problems in economic dynamics.

The theory of monotone concave operators originally due to Krasnoselskiĭ (1964, Chapter 6) is another simple yet powerful tool. The idea behind it is intuitive: imagine a strictly increasing and strictly concave real function f such that $f(x_1) > x_1$ and $f(x_2) < x_2$ with $x_1 < x_2$. Then it must be true that f has a unique fixed point on $[x_1, x_2]$, and by the concavity of f, the fixed point can be computed by successive evaluations of f on any $x \in [x_1, x_2]$. No contraction property is needed here while we still get all the results from the Contraction Mapping Theorem. A full-fledged theorem owing to Du (1989) for arbitrary Banach spaces is stated in Theorem 3.1.

The monotone concave operator theory has seen some recent success in the economic literature. Coleman (1991, 2000) studies the equilibrium in a production economy with income tax and prove the existence and uniqueness of consumption function by constructing a monotone concave map. Following this approach, Datta, Mirman, and Reffett (2002) prove the existence and uniqueness of equilibrium in a large class of dynamic economies with capital and elastic labor supply. More recently, this theory has been applied in asset pricing models with recursive utilities since Marinacci and Montrucchio (2010); other contributions include Borovička and Stachurski (2017, 2018), Becker and Rincon-Zapatero (2017), Marinacci, Montrucchio, et al. (2017), and Bloise and Vailakis (2018). Our work connects to this literature in that the operator which determines the equilibrium price is shown to be increasing and concave but does not satisfy any contraction property. To prove existence and uniqueness, Kikuchi, Nishimura, and Stachurski (2018) use an ad hoc and convoluted method for the case when every firm can only have one upstream partner but fail to generalize it to the multiple partner case. Using the monotone concave operator theory, we are able to extend their results and give a much simpler proof.

Section 2 describes the model in detail. Section 3 introduces the monotone concave operator theory and gives existence and uniqueness results. The algorithm is described in Section 4. Section 5 generalizes the model, allowing for stochastic choices of upstream partners. All proofs can be found in the Appendix.

2 The Model

We study the production chain of a single final good. The chain consists of a single firm at the end of the chain which sells the final good to consumers and firms at different stages of the production, each of which sells an intermediate good to a downstream firm by producing the good in-house or subcontracting a portion of the production process to possibly multiple upstream firms. We index the stage of production by $s \in X = [0, 1]$ with 1 being the final stage. Each firm faces a price function $p : X \to \mathbb{R}_+$ and a cost function $c : X \to \mathbb{R}_+$. Subcontracting incurs a proportionate transaction cost $\delta > 1$ for each upstream partner and an additive transaction cost $g : \mathbb{N} \to \mathbb{R}_+$ which is a function of the number of upstream partners.

We adopt the same assumptions as in Kikuchi, Nishimura, and Stachurski (2018). For the cost function c, we assume that $c(0) = 0$ and it is differentiable, strictly increasing, and strictly convex. In other words, each firm experiences diminishing return to management as mentioned in the introduction. This assumption is needed here because otherwise no firm would want to subcontract its production. We also assume $c'(0) > 0$. For the additive transaction cost function g, we assume that it is strictly increasing, $g(1) = 0$, and $g(k)$ goes to infinity as the number of upstream partners k goes to infinity. To summarize, we have the following two assumptions.

Assumption 2.1. The cost function c is differentiable, strictly increasing, and strictly convex. It also satisfies $c(0) = 0$ and $c'(0) > 0$.

Assumption 2.2. The additive transaction cost function g is strictly increasing and $g(k) \to \infty$ as $k \to \infty$.

Therefore, a firm at stage s solves the following problem:

$$\min_{\substack{t \leq s \\ k \in \mathbb{N}}} \left\{ c(s-t) + g(k) + \delta k p(t/k) \right\}. \tag{1}$$

In (1), the firm chooses to produce $s-t$ in-house with cost $c(s-t)$ and subcontract t to k upstream partners resulting in proportionate transaction cost $\delta k p(t/k)$ and additive transaction cost $g(k)$. Then the firm sells the product to its downstream firm at price $p(s)$.

3 Equilibrium

Following Kikuchi, Nishimura, and Stachurski (2018), in a competitive market with free entry, no firm makes positive profit and the equilibrium price function satisfies

$$p(s) = \min_{\substack{t \leq s \\ k \in \mathbb{N}}} \left\{ c(s-t) + g(k) + \delta k p(t/k) \right\}. \tag{2}$$

Let $R(X)$ be the space of real functions and $C(X)$ the space of continuous functions on X. Then we can define an operator $T : C(X) \to R(X)$ by

$$Tp(s) := \min_{\substack{t \leq s \\ k \in \mathbb{N}}} \left\{ c(s-t) + g(k) + \delta k p(t/k) \right\}. \tag{3}$$

The equilibrium price function is thus determined as the fixed point of the operator T.

3.1 Monotone Concave Operator Theory

Before proceeding to our main result, we first introduce a theorem due to Du (1989), which studies the fixed point properties of monotone concave operators on a partially ordered Banach space.

Let E be a real Banach space on which a partial ordering is defined by a cone $P \subset E$, in the sense that $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ but $x \neq y$, we write $x < y$. An operator $A : E \to E$ is called an *increasing* operator if for all $x, y \in E$, $x \leq y$ implies that $Ax \leq Ay$. It is called a *concave* operator if for any $x, y \in E$ with $x \leq y$ and any $t \in [0, 1]$, we have $A(tx + (1-t)y) \geq tAx + (1-t)Ay$. For any $u_0, v_0 \in E$ with $u_0 < v_0$, we can define an order interval by $[u_0, v_0] := \{x \in E : u_0 \leq x \leq v_0\}$. We have the following theorem (Zhang 2012, Theorem 2.1.2).

Theorem 3.1 (Du–Zhang)**.** *Suppose P is a normal cone*[3]*, $u_0, v_0 \in E$, and $u_0 < v_0$. Moreover, $A : [u_0, v_0] \to E$ is an increasing operator. Let $h_0 = v_0 - u_0$. If A is an concave*

3. A cone $P \subset E$ is said to be normal if there exists $\delta > 0$ such that $\|x + y\| \geq \delta$ for all $x, y \in P$ and $\|x\| = \|y\| = 1$.

operator, $Au_0 \geq u_0 + \epsilon h_0$ for some $\epsilon \in [0, 1]$, and $Av_0 \leq v_0$, then A has a unique fixed point x^* in $[u_0, v_0]$. Furthermore, for any $x_0 \in [u_0, v_0]$, $A^k x_0 \to x^*$ as $k \to \infty$.

This theorem gives a sufficient condition for the existence, uniqueness, and global stability of the fixed point of an operator without assuming it to be a contraction mapping. It is particularly useful in cases where we study a monotone concave operator but the contraction property is hard or impossible to establish. This is the case in our model. The operator T is not a contraction because the transaction cost factor δ is greater than 1, but as will be shown below, T is actually an increasing concave operator. We have the following theorem.

Theorem 3.2. *Let $u_0(s) = c'(0)s$, $v_0(s) = c(s)$, and $[u_0, v_0]$ be the order interval on $C(X)$ with the usual partial order. If Assumption 2.1 and 2.2 hold, then T has a unique fixed point p^* in $[u_0, v_0]$. Furthermore, $T^k p \to p^*$ for any $p \in [u_0, v_0]$.*

This theorem ensures that there exists a unique price function in equilibrium and it can be computed by successive evaluation of the operator T on any function located in that order interval. Furthermore, as is clear in the proof (see Appendix A.1), the existence of the minimizers $t^*(s)$ and $k^*(s)$ can also be proved, although they might not be single valued for some s.

3.2 Properties of the Solution

In the case where each firm can only have one upstream partner, the equilibrium price function is strictly increasing and strictly convex (Kikuchi, Nishimura, and Stachurski 2018). In this model, however, complications arise since firms at different stages might choose to have different numbers of upstream partners. In fact, the equilibrium price is usually piece-wise convex due to this fact. An example[4] of the equilibrium price function is plotted in Figure 1 where $c(s) = e^{10s} - 1$, $g(k) = \beta(k - 1)$ with $\beta = 50$, and $\delta = 10$. As is shown in the plot, the price function as a whole is not convex, but it is piece-wise convex with each piece corresponding to a choice of k. Monotonicity of p^* remains true.

Proposition 3.3. *The equilibrium price function $p^* : X \to \mathbb{R}_+$ is strictly increasing.*

As for comparative statics, we have some basic results about the effect of changing transaction costs on the equilibrium price function. If either transaction cost (δ or g) increases, the equilibrium price function also increases.

4. The parameterization here is merely chosen to highlight the shape of the price function and is not economically realistic. The price is computed using a faster algorithm introduced in Section 4 with $m = 5000$ grid points instead of successive evaluation of T.

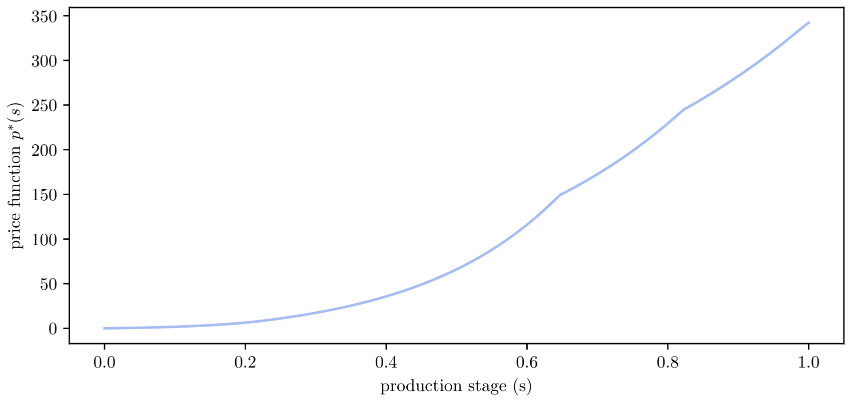


Figure 1: An example of equilibrium price function.

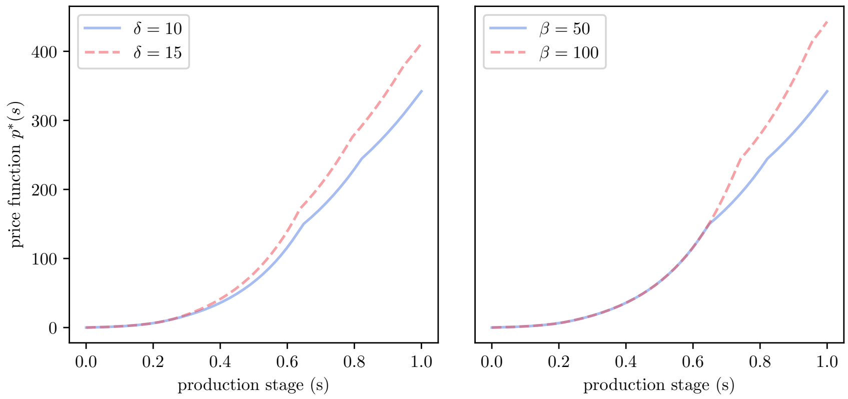


Figure 2: Equilibrium price function when $c(s) = e^{10s} - 1$ and $g(k) = \beta(k - 1)$.

Proposition 3.4. *If $\delta_a \leq \delta_b$, then $p_a^* \leq p_b^*$. Similarly, if $g_a \leq g_b$, $p_a^* \leq p_b^*$.*

In Figure 2, we plot how the equilibrium price function changes when transaction cost increases. The baseline model setting is the same as Figure 1. We can see that if δ or β increases, the equilibrium price function also increases.

4 Computation

To compute an approximation to the equilibrium pricing function for given δ and c, one possibility is to take a function in $[u_0, v_0]$ and iterate with T. However, in practice we can only approximate the iterates, and, since T is not a contraction mapping the rate of convergence can be unsatisfactory for some model settings. On the other hand, as we now show, there is a fast, non-iterative alternative that is guaranteed to converge.

Let $G = \{0, h, 2h, ..., 1\}$ for fixed h. Given G, we define our approximation p to p^* via the recursive procedure in Algorithm 1. In the fourth line, the evaluation of $p(s)$ is by setting

$$p(s) = Tp(s) := \min_{\substack{t \leq s-h \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p(t/k)\}. \tag{4}$$

In line five, the linear interpolation is piecewise linear interpolation of grid points $0, h, 2h, \ldots, s$ and values $p(0), p(h), p(2h), \ldots, p(s)$.

The procedure can be implemented because the minimization step on the right-hand side of (4), which is used to compute $p(s)$, only evaluates p on $[0, s-h]$, and the values of p on this set are determined by previous iterations of the loop. Once the value $p(s)$ has been computed, the following line extends p from $[0, s-h]$ to the new interval $[0, s]$. The process repeats. Once the algorithm completes, the resulting function p is defined on all of $[0, 1]$ and satisfies $p(0) = 0$ and (4) for all $s \in G$ with $s > 0$.

Now consider a sequence of grids $\{G_n\}$, and the corresponding functions $\{p_n\}$ defined by Algorithm 1. Let $G_n = \{0, h_n, 2h_n, \ldots, 1\}$ with $h_n = 2^{-n}$. In this setting we have the following result, the proof of which is given in Appendix A.2.

Theorem 4.1. *If Assumption 2.1 and 2.2 hold, then $\{p_n\}$ converges to p^* uniformly.*

The main advantage of this algorithm is that, for any chosen number of grid points, the number of minimization operations required is fixed, and we can improve the accuracy of this algorithm by increasing the number of grid points. For the iteration method, however, the rate of convergence is different for different model settings and to achieve the same accuracy it usually requires more computation time.

In Figure 3, we plot the computation time of successive iterations of T with $p_0 = c$ (method 1) and Algorithm 1 (method 2) for ten different model settings when the number of grid points is set to be $m = 1000$. The first and last five models are the same[5] except $\delta = 1.1$ for the former and $\delta = 1.01$ for the latter. In each model, we also compute an accurate price function using Algorithm 1 with a very large number of grid points

Algorithm 1 Construction of p from $G = \{0, h, 2h, ..., 1\}$

```
p(0) ← 0
s ← h
while s ≤ 1 do
    evaluate p(s) via equation (4)
    define p on [0, s] by linear interpolation of p(0), p(h), p(2h), ..., p(s)
    s ← s + h
end while
```

5. The cost function c and additive transaction cost function g for the five models are: (1) $c(s) = e^{10s} - 1$, $g(k) = k - 1$; (2) $c(s) = e^s - 1$, $g(k) = 0.01(k-1)$; (3) $c(s) = e^{s^2} - 1$, $g(k) = 0.01(k-1)$; (4) $c(s) = s^2 + s$, $g(k) = 0.01(k-1)$; (5) $c(s) = e^s + s^2 - 1$, $g(k) = 0.05(k-1)$.

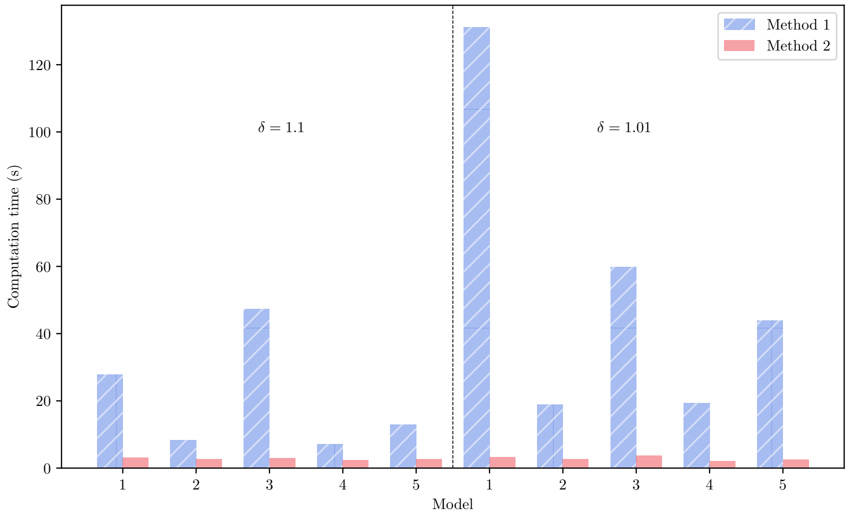


Figure 3: Computation time comparison for the two methods.

($m = 50000$) and compare it with results from both methods when $m = 1000$. We find that the error from method 2 is comparable or smaller than that from method 1 in each model. The algorithm achieves more accurate results at a much faster speed. As we can see in Figure 3, method 2 completes the computation in around 3 seconds in each model while the computation time of method 1 ranges from 7 seconds to more than 2 minutes. The speed difference is especially drastic when δ is close to 1, since it takes T more iterations to converge with smaller δ but the number of operations for the algorithm is fixed. In model 1 with $\delta = 1.01$, the algorithm is 40 times faster than successive iterations of T!

5 Stochastic Choices

So far we have discussed the case where each firm can choose the optimal number of upstream partners according to (1). In reality, however, firms usually face uncertainty when choosing their partners. The result is that some firms might choose fewer or more partners than what is optimal. For instance, a firm might not be able to choose a certain number of upstream partners due to regulation or failure to arrive at agreements with potential partners. Conversely, the upstream partners of a firm might experience supply shocks and fail to meet production requirements, causing it to sign more partners than what is optimal and bear more transaction costs. In this section, we model this scenario and incorporate uncertainty into each firm's optimization problem.

We assume that each firm chooses an amount of "search effort" λ and the resulting number of upstream partners follows a Poisson distribution[6] with parameter λ that starts

6. Note that in the usual sense, if a random variable X follows the Poisson distribution, X takes values in nonnegative integers. Here we shift the probability function so that k starts from 1.

from $k = 1$. In other words, the probability of having k partners is

$$f(k; \lambda) = \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!}$$

when $\lambda > 0$. We also assume that when $\lambda = 0$, $\text{Prob}(n = 1) = 1$, that is, each firm can always choose to have only one upstream partner with certainty. For example, if a firm chooses to exert effort $\lambda = 2.5$, the probabilities of it ending up with 1, 2, 3, 4, 5 partners are, respectively: 0.08, 0.2, 0.26, 0.21, 0.13. One characteristic of the Poisson distribution is that both its mean and variance increase with λ, which makes it suitable for our model since the more partners a firm aims for, the more uncertainty there will be in the contracting process.

Hence, a firm at stage s solves the following problem:

$$\min_{\substack{t \leq s \\ \lambda \geq 0}} \left\{ c(s-t) + \mathbb{E}_k^\lambda \left[g(k) + \delta k p(t/k) \right] \right\} \tag{5}$$

where $\mathbb{E}_k^\lambda$ stands for taking expectation of k under the Poisson distribution with parameter λ. Specifically,

$$\mathbb{E}_k^\lambda \left[g(k) + \delta k p(t/k) \right] = \sum_{k=1}^{\infty} \left[g(k) + \delta k p(t/k) \right] f(k; \lambda).$$

Similar to Section 3, we can define another operator $\tilde{T} : C(X) \to R(X)$ by

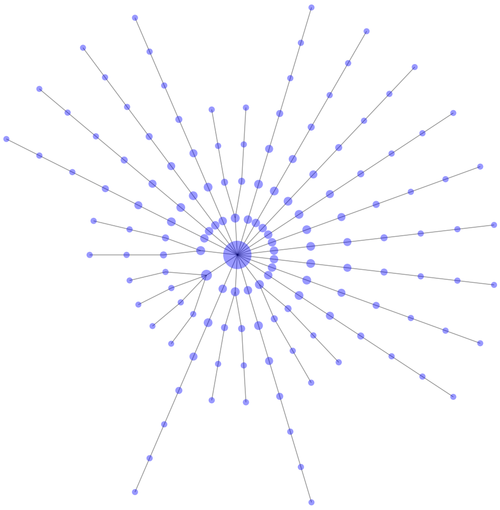
$$\tilde{T} p(s) := \min_{\substack{t \leq s \\ \lambda \geq 0}} \left\{ c(s-t) + \mathbb{E}_k^\lambda \left[g(k) + \delta k p(t/k) \right] \right\}. \tag{6}$$

As will be shown in Appendix A.3, all of the above results still apply in the stochastic case and we summarize them in the following theorem.

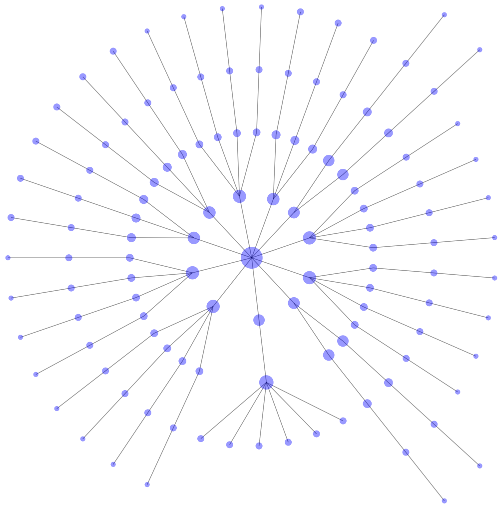
Theorem 5.1. *Let $u_0(s) = c'(0)s$, $v_0(s) = c(s)$. If Assumption 2.1 and 2.2 hold, then the operator $\tilde{T}$ has a unique fixed point $\tilde{p}^*$ in $[u_0, v_0]$ and $\tilde{T}^k p \to \tilde{p}^*$ for any $p \in [u_0, v_0]$. Furthermore, $\tilde{p}_n$ from Algorithm 1 converges to $\tilde{p}^*$ uniformly.*

By Theorem 5.1, there exists a unique equilibrium price function $\tilde{p}^*$ and we can compute it either by successive evaluation of $\tilde{T}$ or by Algorithm 1. The algorithm is particularly useful here since it now takes much longer time to complete one minimization operation with firms choosing continuous values of λ instead of discreet values of k.

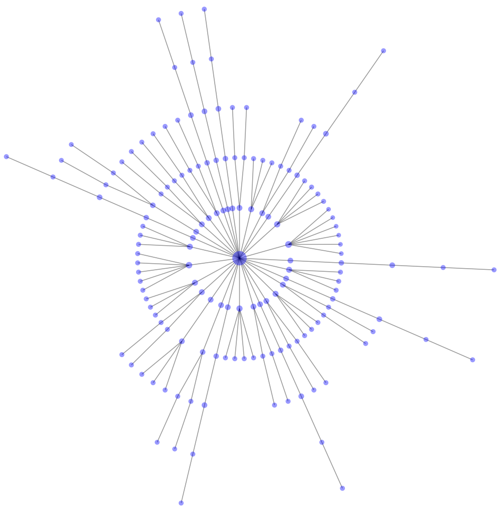
Similarly, there exist minimizers t^* and λ^* so that firm at any stage s has an optimal choice $t^*(s)$ and $\lambda^*(s)$. With the optimal choice functions, we can compute an equilibrium firm allocation recursively as in Kikuchi, Nishimura, and Stachurski (2018). Specifically, we start at the most downstream firm at $s = 1$ and compute its optimal choices t^* and λ^*. Next, we pick a realization of k according to the Poisson distribution with parameter λ^* and repeat the process for each of its upstream firm at $s' = t^*/k$. The whole process ends



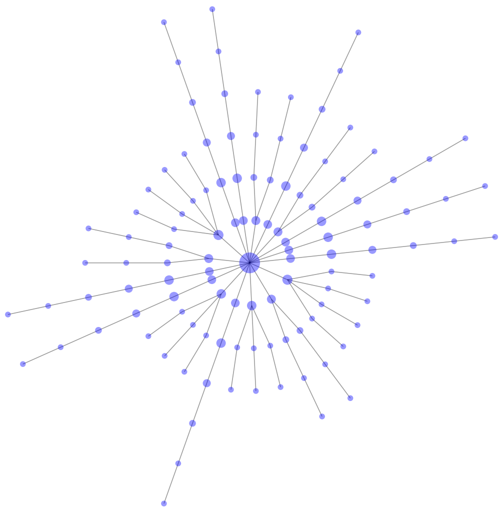
(a) $\beta = 0.0005$, $\delta = 1.05$, $\theta = 1.2$



(b) $\beta = 0.0005$, $\delta = 1.1$, $\theta = 1.2$



(c) $\beta = 0.0001$, $\delta = 1.05$, $\theta = 1.2$



(d) $\beta = 0.0005$, $\delta = 1.05$, $\theta = 1.15$

Figure 4: Production network with stochastic choices of upstream partners

when all the most upstream firms choose to carry out the remaining production process by themselves. Note that due to the stochastic nature of this model, each simulation will give a different firm allocation.

In Figure 4, we plot some production networks for different model parameterizations using the above approach. Each circle represents a firm and the one at the center is the firm at $s = 1$. The diameter of each circle is proportionate to the size[7] of the corresponding firm. The cost function is set to be $c(s) = s^\theta$ and the additive transaction cost is $g(k) = \beta k^{1.5}$. Compared with production networks in Kikuchi, Nishimura, and Stachurski (2018), the

7. Here the firm size is calculated using its value added $c(s - t^*) + g(k)$ where k is a realization of the Poisson distribution with parameter λ^*.

graphs here are no longer symmetric since even firms on the same layer can have different realized numbers of upstream partners and thus different firm sizes.

Comparing (a) and (b), an increase in transaction cost makes firms in (b) outsource less and produce more in-house, resulting in fewer layers in the production network. Similarly, comparing (c) with (a), a decrease in additive transaction costs encourages firms at each level to find more subcontractors. The results are more but smaller firms at each level and fewer layers in the network. Comparing (d) with (a), the difference is a decrease in curvature of the cost function c, which makes outsourcing less appealing. The firms in (d) tend to produce more in-house, resulting in a production network of fewer layers.

A Appendix

A.1 Proofs from Section 3

Let $U = \mathbb{N} \times [0, 1]$ equipped with the Euclidean metric in $\mathbb{R}^2$ and X be equipped with the Euclidean metric in $\mathbb{R}$. To simplify notation, we can write T as

$$Tp(s) = \min_{(k,t)\in\Theta(s)} f_p(s, k, t)$$

where $\Theta : X \to U$ is a correspondence defined by $\Theta(s) = \mathbb{N} \times [0, s]$, and $f_p(s, k, t) = c(s - t) + g(k) + \delta kp(t/k)$.

Lemma A.1. *$Tp \in C([0, 1])$ for all $p \in C([0, 1])$.*

Proof. We use Berge's theorem to prove continuity. By Assumption 2.2, we can restrict Θ to be $\Theta(s) = \{1, 2, \dots, \bar{k}\} \times [0, s]$ for some large $\bar{k} \in \mathbb{N}$. Then Θ is compact-valued.

To see Θ is upper hemicontinuous, note $\Theta(s)$ is closed for all $s \in X$. Since the graph of Θ is also closed, by the Closed Graph Theorem (see, e.g., Aliprantis and Border (2006, p. 565)), Θ is upper hemicontinuous on X.

To check for lower hemicontinuity, fix $s \in X$. Let V be any open set intersecting $\Theta(s) = \{1, 2, \dots, \bar{k}\} \times [0, s]$. Then it is easy to see that we can find a small $\epsilon > 0$ such that $\Theta(s') \cap V \neq \emptyset$ for all $s' \in [s - \epsilon, s + \epsilon]$. Hence Θ is lower hemicontinuous on X.

Because $p \in C([0, 1])$, f_p is jointly continuous in its three arguments. By Berge's theorem, Tp is continuous on X. □

Note that by Berge's theorem, the minimizers t^* and k^* exist and are upper hemicontinuous.

Lemma A.2. *T is increasing and concave.*

Proof. It is apparent that T is increasing. To see T is concave, let $p, q \in C([0,1])$ and $\alpha \in (0,1)$. Then we have

$$\begin{aligned}\alpha Tp(s) + (1-\alpha)Tq(s) &= \min_{(k,t)\in\Theta(s)} \alpha f_p(s,k,t) + \min_{(k,t)\in\Theta(s)} (1-\alpha) f_q(s,k,t) \\ &\leq \min_{(k,t)\in\Theta(s)} \{\alpha f_p(s,k,t) + (1-\alpha) f_q(s,k,t)\} \\ &= \min_{(k,t)\in\Theta(s)} \{c(s-t) + g(k) + \delta k\left[\alpha p(t/k) + (1-\alpha) q(t/k)\right]\} \\ &= \min_{(k,t)\in\Theta(s)} f_{\alpha p + (1-\alpha) q}(s,k,t) \\ &= T\left[\alpha p + (1-\alpha) q\right](s)\end{aligned}$$

which completes the proof. □

Lemma A.3. *$Tu_0 \geq u_0 + \epsilon(v_0 - u_0)$ for some $\epsilon \in (0,1)$.*

Proof. Define $\bar{s} := \max\{0 \leq s \leq 1 : c'(s) \leq \delta c'(0)\}$. Then we have

$$\begin{aligned}Tu_0(s) &= \min_{(k,t)\in\Theta(s)} f_{u_0}(s,k,t) \\ &= \min_{(k,t)\in\Theta(s)} \{c(s-t) + g(k) + \delta c'(0) t\} \\ &= \min_{t\leq s} \{c(s-t) + \delta c'(0) t\} \\ &= \begin{cases} c(\bar{s}) + \delta c'(0)(s-\bar{s}), & \text{if } s \geq \bar{s} \\ c(s), & \text{if } s < \bar{s} \end{cases}\end{aligned}$$

Since $Tu_0(s) > u_0(s)$ for all s except at 0, we can find $\epsilon \in (0,1)$ such that $Tu_0 \geq u_0 + \epsilon(v_0 - u_0)$. □

Lemma A.4. *$Tv_0 \leq v_0$.*

Proof. Choose $k = 1$ and $t = 0$. We have $Tv_0(s) \leq c(s-0) + g(1) + \delta c(0) = c(s) = v_0(s)$. □

Proof of Theorem 3.2. Since $P = \{f \in C(X) : f(x) \geq 0 \text{ for all } x \in X\}$ is a normal cone, the theorem follows from the previous lemmas and Theorem 3.1. □

Proof of Proposition 3.3. We first show that T maps a strictly increasing function to a strictly increasing function. Suppose $p \in [u_0, v_0]$ and is strictly increasing. Pick any $s_1, s_2 \in [0,1]$ with $s_1 < s_2$. Let t^* and k^* be the minimizers of T. To simplify notation, let $t_1 \in t^*(s_1)$, $t_2 \in t^*(s_2)$, $k_1 \in k^*(s_1)$, and $k_2 \in k^*(s_2)$. If $t_2 \leq s_1$, then we have

$$\begin{aligned}Tp(s_2) &= c(s_2 - t_2) + g(k_2) + \delta k_2 p(t_2/k_2) \\ &> c(s_1 - t_2) + g(k_2) + \delta k_2 p(t_2/k_2) \\ &\geq Tp(s_1).\end{aligned}$$

If $s_1 < t_2 \leq s_2$, then $t_2 + s_1 - s_2 \leq s_1$. Since p is strictly increasing, we have

$$\begin{aligned} Tp(s_2) &= c\left(s_1 - (t_2 + s_1 - s_2)\right) + g(k_2) + \delta k_2 p(t_2/k_2) \\ &> c\left(s_1 - (t_2 + s_1 - s_2)\right) + g(k_2) + \delta k_2 p\left((t_2 + s_1 - s_2)/k_2\right) \\ &\geq Tp(s_1). \end{aligned}$$

Since $c \in [u_0, v_0]$, by Theorem 3.2, $T^k c \to p^*$ as $k \to \infty$. Furthermore, since c is strictly increasing, it follows from the above result that p^* is strictly increasing. □

Proof of Proposition 3.4. If $\delta_a \leq \delta_b$, then $T_a p \leq T_b p$ for any $p \in [u_0, v_0]$. Since T is increasing by Lemma A.2, we have $T_a^k p \leq T_b^k p$ for any $p \in [u_0, v_0]$ and any $k \in \mathbb{N}$. Then by Theorem 3.2, $p_a^* \leq p_b^*$. The same arguments applies if $g_a \leq g_b$. □

A.2 Proof of Theorem 4.1

Lemma A.5. *The function p_n is increasing for every n.*

Proof. As p_n is piecewise linear, we shall prove it by induction. Since $p_n(0) = 0$ and $p_n(h_n) = c(h_n)$, p_n is increasing on $[0, h_n]$. Suppose it is increasing on $[0, s]$ for some $s = h_n, 2h_n, \ldots, 1 - h_n$, then we have

$$\begin{aligned} p_n(s + h_n) &= \min_{t \leq s,\, k \in \mathbb{N}} \{c(s + h_n - t) + g(k) + \delta k p_n(t/k)\} \\ &= c(s + h_n - t^*) + g(k^*) + \delta k^* p_n(t^*/k^*) \end{aligned}$$

where t^* and k^* are the minimizers. If $t^* \leq s - h_n$, it follows from the monotonicity of c that

$$\begin{aligned} p_n(s + h_n) &\geq c(s - t^*) + g(k^*) + \delta k^* p_n(t^*/k^*) \\ &\geq \min_{t \leq s - h_n,\, k \in \mathbb{N}} \{c(s - t) + g(k) + \delta k p_n(t/k)\} \\ &= p_n(s). \end{aligned}$$

If $t^* \in (s - h_n, s]$, then $s + h_n - t^* \geq h_n$. Because p_n is increasing on $[0, s]$, we have

$$\begin{aligned} p_n(s + h_n) &\geq c[s - (s - h_n)] + g(k^*) + \delta k^* p_n[(s - h_n)/k^*] \\ &\geq \min_{t \leq s - h_n,\, k \in \mathbb{N}} \{c(s - t) + g(k) + \delta k p_n(t/k)\} \\ &= p_n(s), \end{aligned}$$

which completes the proof. □

Lemma A.6. *The sequence $\{p_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous.*

Proof. To see $\{p_n\}$ is uniformly bounded, note that for each n,

$$\begin{aligned} p_n(s+h_n) &= \min_{t\le s,\, k\in\mathbb{N}} \{c(s+h_n-t)+g(k)+\delta k p_n(t/k)\} \\ &\le c(s+h_n)+g(1)+\delta p_n(0) \\ &= c(s+h_n) \le c(1) \end{aligned}$$

for all $s = 0, h_n, \ldots, 1-h_n$.

Due to Lemma A.5, to see $\{p_n\}$ is equicontinuous, it suffices to show that there exists $K > 0$ such that $p_n(s+h_n) - p_n(s) \le K h_n$ for all $n \in \mathbb{N}$ and all $s = 0, h_n, 2h_n, \ldots, 1-h_n$. Fix such n and s. If $s = 0$, $p_n(h_n) - p_n(0) = c(h_n) \le c'(1)h_n$. If $s \ge h_n$, denote the minimizers in the definition of $p_n(s)$ by t^* and k^*, i.e.,

$$\begin{aligned} p_n(s) &= \min_{t\le s-h_n,\, k\in\mathbb{N}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} \\ &= c(s-t^*)+g(k^*)+\delta k^* p_n(t^*/k^*). \end{aligned}$$

Since $t^* \le s$, it follows that

$$\begin{aligned} p_n(s+h_n) &= \min_{t\le s,\, k\in\mathbb{N}} \{c(s+h_n-t)+g(k)+\delta k p_n(t/k)\} \\ &\le c(s+h_n-t^*)+g(k^*)+\delta k^* p_n(t^*/k^*). \end{aligned}$$

Hence,

$$\begin{aligned} p_n(s+h_n)-p_n(s) &\le c(s+h_n-t^*)+c(s-t^*) \\ &\le c'(1)h_n, \end{aligned}$$

which completes the proof. □

Lemma A.7. *There exists a uniformly convergent subsequence of $\{p_n\}$. Furthermore, every uniformly convergent subsequence of $\{p_n\}$ converges to a fixed point of T.*

Proof. Lemma A.6 and the Arzelà-Ascoli theorem implies that p_n has a uniformly convergent subsequence. To simplify notation, let $\{p_n\}$ be such a subsequence and converge uniformly to $\bar{p}$. Because p_n are continuous, $\bar{p}$ is continuous. By Berge's theorem,

$$T\bar{p}(s) = \min_{t\le s,\, k\in\mathbb{N}} \{c(s-t)+g(k)+\delta k \bar{p}(t/k)\}$$

is also continuous. To see $\bar{p}$ is a fixed point of T, it is sufficient to show that $\bar{p}$ and $T\bar{p}$ agree on the dyadic rationals $\cup_n G_n$, i.e.,

$$\lim_{n\to\infty} \min_{\substack{t\le s-h_n \\ k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} = \min_{t\le s,\, k\in\mathbb{N}} \{c(s-t)+g(k)+\delta k \bar{p}(t/k)\}$$

for every $s \in \cup_n G_n$.

Fix $\epsilon > 0$. Since $p_n \to \bar{p}$ uniformly, there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that

$$p_n(x) > \bar{p}(x) - \epsilon/(\delta \bar{k})$$

for all $x \in [0,1]$ where $\bar{k}$ is the upper bound on the possible values of k. It follows that for $n > N_1$ we have

$$\begin{aligned}\min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} &> \min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\} - \epsilon \\ &\ge \min_{\substack{t \le s \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\} - \epsilon.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} \ge \min_{t \le s,\ k \in \mathbb{N}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\}.$$

For the other direction, there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that

$$p_n(x) < \bar{p}(x) + \epsilon/(2\delta \bar{k})$$

for all $x \in [0,1]$. Then for $n > N_2$ we have

$$\min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} < \min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\} + \epsilon/2.$$

Since $c, g, \bar{p}$ are continuous and $h_n \to 0$, we can choose N_3 such that $n > N_3$ implies that

$$\min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\} < \min_{\substack{t \le s \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\} + \epsilon/2.$$

Hence, for $n > \max\{N_2, N_3\}$ we have

$$\min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} < \min_{\substack{t \le s \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\} + \epsilon.$$

This implies

$$\lim_{n\to\infty} \min_{\substack{t \le s - h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} \le \min_{t \le s,\ k \in \mathbb{N}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\}.$$

Therefore, $\bar{p} = T\bar{p}$. □

Lemma A.8. *Every uniformly convergent subsequence of $\{p_n\}$ converges to p^*.*

Proof. Let $\{p_n\}$ be the subsequence that converges uniformly to $\bar{p}$. By Theorem 3.2, to see $\bar{p} = p^*$, it suffices to show that $\bar{p}$ is continuous and $c'(0)x \le \bar{p}(x) \le c(x)$ for all $x \in [0,1]$. Continuity is satisfied by the fact that each p_n is continuous and $p_n \to \bar{p}$ uniformly. To show the second one, we again prove this holds on $\cup_n G_n$, and it is sufficient to show that $c'(0)s \le p_n(s) \le c(s)$ for all $s \in G_n$ and all $n \in \mathbb{N}$. It is apparent that $p_n(s) \le c(s)$ (choose $t = 0$ and $k = 1$). We show $p_n(s) \ge c'(0)s$ by induction. Suppose $p_n(x) \ge c'(0)x$

for all $x \leq s$. Then we have

$$\begin{aligned} p_n(s+h_n) &= \min_{t\leq s,\, k\in\mathbb{N}} \{c(s+h_n-t)+g(k)+\delta k p_n(t/k)\} \\ &\geq \min_{t\leq s,\, k\in\mathbb{N}} \{c'(0)(s+h_n-t)+g(k)+\delta c'(0)t\} \\ &= \min_{t\leq s} \{c'(0)(s+h_n-t+\delta t)\} \\ &= c'(0)(s+h_n). \end{aligned}$$

Since $p_n(0) = 0 \geq c'(0)\cdot 0$, it follows that $p_n(s) \geq c'(0)s$. This concludes the proof. □

A.3 Proof of Theorem 5.1

Similar to Appendix A.1, we can write the operator $\tilde{T}$ in (6) as

$$\tilde{T}p(s) = \min_{(\lambda,t)\in\tilde{\Theta}(s)} \left\{c(s-t)+\mathbb{E}_k^\lambda\left[g(k)+\delta k p(t/k)\right]\right\}$$

where $\tilde{\Theta}(s) = [0,\infty)\times[0,s]$. Upon close inspection, all of the above lemmas still hold for $\tilde{T}$ if we can restrict $\tilde{\Theta}(s)$ to be a compact set. To be more specific, Lemma A.2 and A.5 can be proved in the exact same way; Lemma A.3, A.4, A.6, and A.8 hold since each firm can choose $k=1$ with probability 1; Lemma A.7 and A.1 need the compactness of $\tilde{\Theta}(s)$. To avoid redundancy, we omit the proofs and shall only show that there exists an upper bound on the choice set of λ.

Let ν be the median of the Poisson distribution and denote the ceiling of ν (i.e., the least integer greater than or equal to ν) by $\bar{\nu}$. Then we have

$$\sum_{k=\bar{\nu}}^{\infty} f(k;\lambda) \geq \frac{1}{2}$$

by definition. It follows that the expectation of $g(k)$

$$\begin{aligned} \mathbb{E}_k^\lambda g(k) &= \sum_{k=1}^{\infty} g(k)f(k;\lambda) \\ &\geq \sum_{k=\bar{\nu}}^{\infty} g(k)f(k;\lambda) \\ &\geq g(\bar{\nu})\sum_{k=\bar{\nu}}^{\infty} f(k;\lambda) \\ &\geq \frac{1}{2}g(\bar{\nu}) \end{aligned}$$

where the second inequality follows from Assumption 2.2. Choi (1994) gives bounds[8] for

8. Since in our model k starts from 1, we write $\nu-1$ in the inequality.

the median of the Poisson distribution:

$$\lambda - \ln 2 \le \nu - 1 < \lambda + \frac{1}{3}.$$

So we have

$$\mathbb{E}_k^\lambda g(k) \ge \frac{1}{2} g(\bar{\nu}) \ge \frac{1}{2} g(\nu) \ge \frac{1}{2} g(\lambda - \ln 2 + 1).$$

Therefore, we can find $\bar{\lambda}$ such that $\mathbb{E}_k^\lambda g(k) \ge c(1)$ for all $\lambda \ge \bar{\lambda}$ and hence $\Theta(s)$ is essentially $[0, \bar{\lambda}] \times [0, s]$ which is a compact set.

References

Acemoglu, Daron, Vasco M Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. 2012. "The network origins of aggregate fluctuations." *Econometrica* 80 (5): 1977–2016.

Acemoglu, Daron, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. 2015a. *Networks, shocks, and systemic risk.* Technical report. National Bureau of Economic Research.

———. 2015b. "Systemic risk and stability in financial networks." *American Economic Review* 105 (2): 564–608.

Aliprantis, Charalambos D, and Kim C Border. 2006. "Infinite dimensional analysis: a hitchhiker's guide."

Baldwin, Richard, and Anthony J Venables. 2013. "Spiders and snakes: offshoring and agglomeration in the global economy." *Journal of International Economics* 90 (2): 245–254.

Becker, Gary S, and Kevin M Murphy. 1992. "The division of labor, coordination costs, and knowledge." *The Quarterly Journal of Economics* 107 (4): 1137–1160.

Becker, Robert A, and Juan P Rincon-Zapatero. 2017. "Recursive Utiity and Thompson Aggregators."

Bellman, Richard Ernest. 1957. "Dynamic Programming."

Bigio, Saki, and Jennifer La'O. 2016. *Financial frictions in production networks.* Technical report. National Bureau of Economic Research.

Bloise, Gaetano, and Yiannis Vailakis. 2018. "Convex dynamic programming with (bounded) recursive utility." *Journal of Economic Theory* 173:118–141.

Borovička, Jaroslav, and John Stachurski. 2017. *Necessary and Sufficient Conditions for Existence and Uniqueness of Recursive Utilities.* Technical report. National Bureau of Economic Research.

Borovička, Jaroslav, and John Stachurski. 2018. *Existence and Uniqueness of Equilibrium Asset Prices over Infinite Horizons.* Technical report.

Carvalho, Vasco. 2007. "Aggregate fluctuations and the network structure of intersectoral trade."

Choi, Kwok Pui. 1994. "On the medians of gamma distributions and an equation of Ramanujan." *Proceedings of the American Mathematical Society* 121 (1): 245–251.

Ciccone, Antonio. 2002. "Input chains and industrialization." *The Review of Economic Studies* 69 (3): 565–587.

Coleman, Wilbur John. 1991. "Equilibrium in a production economy with an income tax." *Econometrica: Journal of the Econometric Society:* 1091–1104.

———. 2000. "Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities." *Journal of Economic Theory* 95 (1): 71–78.

Datta, Manjira, Leonard J Mirman, and Kevin L Reffett. 2002. "Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor." *Journal of Economic Theory* 103 (2): 377–410.

Du, Yihong. 1989. "Fixed points of a class of non-compact operators and applications" [in Chinese]. *Acta Mathematica Sinica* 32 (5): 618–627.

Jones, Charles I. 2011. "Intermediate goods and weak links in the theory of economic development." *American Economic Journal: Macroeconomics* 3 (2): 1–28.

Kikuchi, Tomoo, Kazuo Nishimura, and John Stachurski. 2018. "Span of control, transaction costs, and the structure of production chains." *Theoretical Economics* 13 (2): 729–760.

Krasnoselskiĭ. 1964. *Positive Solutions of Operator Equations.*

Levine, David K. 2012. "Production chains." *Review of Economic Dynamics* 15 (3): 271–282.

Lucas, Robert E. 1978. "On the size distribution of business firms." *The Bell Journal of Economics:* 508–523.

Marinacci, Massimo, and Luigi Montrucchio. 2010. "Unique solutions for stochastic recursive utilities." *Journal of Economic Theory* 145 (5): 1776–1804.

Marinacci, Massimo, Luigi Montrucchio, et al. 2017. *Unique tarski fixed points.* Technical report.

Zhang, Zhitao. 2012. *Variational, topological, and partial order methods with their applications.* Vol. 29. Springer Science & Business Media.