Herding driven by the desire to differ

Sander Heinsalu*

June 26, 2019

Abstract

Observational learning often involves congestion: an agent gets less payoff from an action when more predecessors have taken that action. This preference to act differently from previous agents may paradoxically increase all but one agent's probability of matching the actions of the predecessors. The reason is that when previous agents conform to their predecessors despite the preference to differ, their actions become more informative. The desire to match predecessors' actions may reduce herding by a similar reasoning.

Keywords: herding, information cascade, social preferences, congestion.

JEL classification: D83, D82, C73, C72.

This paper studies rational agents' learning from the choices of others when the information of others is not directly available. Payoffs are interdependent due to congestion costs: if more preceding agents choose an action, then an agent's payoff from taking that action falls. Congestion arises in many situations, such as when individuals choose a supermarket lane or other queue, a route to drive or a service provider to use. For firms, choosing a market that others have entered is less profitable, other things equal.

The model follows the seminal paper of Bikhchandani et al. (1992). Agents choose in sequence between two actions, after observing the previous agents' actions and a private signal. All agents prefer their action to match a binary state, which is symmetrically unknown. The

*Research School of Economics, Australian National University, 25a Kingsley St, Acton ACT 2601, Australia. Email: sander.heinsalu@anu.edu.au, website: https://sanderheinsalu.com/ The author thanks George Mailath, Rakesh Vohra, Andrew Postlewaite and the audiences at the University of Warwick and the conference Pressing and Challenging Changes for comments and suggestions. The author is grateful to the University of Pennsylvania for its hospitality during part of this work.

payoff of an agent also increases when the action differs from those of the preceding agents. Such social preferences are also assumed in Gaigl (2009) and Eyster et al. (2014).

In equilibrium, the preference for an action different from that of previous agents may paradoxically increase all but one agent's probability of matching the actions of the predecessors, compared to the case when payoffs do not depend on others' actions. When previous agents choose the same action as their predecessors, congestion costs increase the informativeness of this action. The reason is that a stronger signal is required to induce an action when the preceding agents have chosen it. A more informative action in turn motivates imitation, even when congestion moderately increases the cost of imitating.

Similarly, the desire to conform to previous agents' actions may reduce herding. If past agents made the same choice as their predecessors, then these actions are less informative under a preference to match previous moves. The decreased informativeness of preceding actions allows an agent's private signal to outweigh the combined effect of previous moves and the desire to conform.

In contrast to the current work, both Gaigl (2009) and Eyster et al. (2014) show that congestion costs reduce herding and, if not too large, improve learning. Large enough congestion costs cause agents to alternate their actions (anti-herd), which decreases learning. Gaigl (2009) and Eyster et al. (2014) focus on asymptotic learning, but the present paper considers the probability of the first n agents matching the actions of their predecessors, as well as choosing the correct action. When the desire to differ is small enough, all agents take the same action from some finite time onward, as in the previous literature. In that case, learning is bounded, i.e. there is positive probability of the wrong action as time goes to infinity.

? show the possibility of herding in financial markets where price responses create a desire to differ from previous transactions. The preference to mismatch preceding traders decreases the probability of herds, unlike in the current paper.

Eyster and Rabin (2014) can be interpreted as comparative statics similar to this paper, but operating via the observation structure instead of preferences. Suppose agents 1 and 2 take the same action. If agent 3 believes that 2 did not observe 1, then 3 is more likely to imitate 1 and 2 than when 3 believes 2 observed 1, because coinciding independent actions of 1 and 2 are more informative than coinciding actions when 2 observes and potentially mimicks 1. In the current work, if 3 believes that 2 prefers to differ from 1, then 3 is more likely to imitate 1 and 2 than when 2 does not care about 1's action, because coinciding actions are more informative under a desire to differ. Correlation neglect, as in Eyster and

Rabin (2010), has a similar effect on actions as a congestion cost, but the welfare implications differ. If agent 3 neglects that 2 sometimes imitates 1, then 3 treats coinciding actions by 1 and 2 as a stronger signal than a Bayesian and herds more often. In the current work, 3 correctly infers that 2 mimicking 1 is a stronger signal when 2 desires to differ from 1. Under correlation neglect, observing others may reduce payoffs relative to not observing, unlike in the present paper.

In Callander and Hörner (2009), agents have different precisions of information and observe only the number of times an action was chosen, not who chose it. Following the minority is optimal when the agents with more precise information are sufficiently numerous and better-informed.

The desire to conform (Appendix A) may be interpreted as the persuasion bias of DeMarzo et al. (2003)—beliefs move in the direction of redundant messages. Interpret previous agents' actions as messages in favour of those actions. Both over-inferring from redundant information and conformism motivate an individual to match previous actions. Unlike in DeMarzo et al. (2003), in the present paper agents know the persuasion bias of others. This knowledge more than outweighs each agent's own bias, as seen from the reduced herding, which contrasts with DeMarzo et al. (2003) where all agents converge to the same belief.

Other forms of social preference in herding have been studied. In Ali and Kartik (2012), agents prefer others to take the correct action. Callander (2007) assumes that agents want to match the eventual majority, thus payoff depends on future agents' choices, unlike in the current work. In the pathbreaking papers of Banerjee (1992); Bikhchandani et al. (1992), payoffs are independent of the actions of others and the action is chosen from a continuum.

The next section sets up the model where agents desire to differ from previous movers. The results are collected in Section 2 and discussed in Section 7. The appendix shows that a desire to conform may reduce herding.

1 Social learning with a desire to differ

Time is discrete, with periods and players indexed by $i \in \mathbb{N}$. In period i, player i observes a private signal $s_i \in \{\textsc{l}, \ell, r, \textsc{r}\}$ and chooses a public action $a_i \in \{L, R\}$. The public history of actions up to time t is denoted $a^t = (a_1, \ldots, a_t)$. Action a_i is called *uninformative* after history a^{i-1} if $a_i(a^{i-1}, s_i)$ is constant in s_i, and *informative* otherwise. *Herding* after history a^t means that $a_{t+1} = a_t$ regardless of signals. *Anti-herding* means $a_{t+1} \neq a_t$ regardless of signals.

An unknown state $\theta \in \{\mathcal{L}, \mathcal{R}\}$ determines payoffs via

$$u_i(a^i, \theta) = \mathbf{1}\{a_i = \theta\} - \frac{k}{i-1}\sum_{j=1}^{i-1}\mathbf{1}\{a_j = a_i\},$$

where $\mathbf{1}S$ denotes the indicator function of set S and $k \geq 0$ is the congestion cost. If $k = 0$, then the environment is standard herding with independent preferences. If $k > 0$, then each player prefers to take a different choice than the majority of the previous agents, other things equal.

The prior probability of state $\mathcal{R}$ is $p_0 \in [\frac{1}{2}, 1)$ w.l.o.g. Denote by $p_S \in (0,1)$ the unconditional probability of signal $s_i \in \{\text{L}, \text{R}\}$. Conditional on the state, the probabilities of the signals are $\Pr(\text{L}|\mathcal{L}) = \Pr(\text{R}|\mathcal{R}) = Q \in \left(\frac{p_S}{2}, p_S\right)$ and $\Pr(\ell|\mathcal{L}) = \Pr(r|\mathcal{R}) = q \in \left(\frac{1-p_S}{2}, \frac{Q(1-p_S)}{p_S}\right)$. Therefore $\Pr(\text{L}|\mathcal{R}) = \Pr(\text{R}|\mathcal{L}) = p_S - Q$ and $\Pr(\ell|\mathcal{R}) = \Pr(r|\mathcal{L}) = 1 - p_S - q$.

Bayesian updating determines each player's posterior belief $p_i = \Pr(\mathcal{R}|a^{i-1}, s_i)$ and log likelihood ratio $l_i := \ln p_i - \ln(1-p_i)$. Using l_i instead of p_i simplifies the exposition and is mathematically equivalent. Signals L, ℓ favour state $\mathcal{L}$, in the sense of increasing the posterior probability of $\mathcal{L}$. Similarly, R, r favour state $\mathcal{R}$. Calling signals ℓ, r *weak* and L, R *strong* is justified by $\frac{q}{1-p_S} < \frac{Q}{p_S}$, which means that the posterior belief moves more in response to L, R than to ℓ, r. Assume $q > p_0(1-p_S)$, equivalently $l_q > l_0$, to ensure signals are informative enough for even a weak signal $s \in \{\ell, r\}$ to overturn the prior, i.e. player 1 to believe after signal ℓ that state $\mathcal{L}$ is more likely than $\mathcal{R}$.

Denote the (public) log likelihood ratio of player $i > 1$ before observing s_i by $l_i(a^{i-1})$. To derive player i's private log likelihood ratios $l_i(a^{i-1}, s_i)$ after a^{i-1}, s_i, define

$l_Q := \ln Q - \ln(p_S - Q) > 0$,

$l_q := \ln q - \ln(1 - p_S - q) \in (0, l_Q)$,

$l_{Qq} := \ln(Q+q) - \ln(1-Q-q) \in (l_q, l_Q)$ and

$l_{\neg Q} := \ln(1 - p_S + Q) - \ln(1-Q) \in (0, l_{Qq})$,

where l_Q, l_q are the log likelihood ratios of strong and weak signals respectively. The log likelihood ratio l_{Qq} does not distinguish strong and weak signals, only whether the signal favours $\mathcal{L}$ or $\mathcal{R}$. If the strong signal in favour of one state is distinguishable from the other three, but the latter look identical to an agent, then upon not seeing the distinguishable strong signal, the agent uses $l_{\neg Q}$ to update. The (private) log likelihood ratios of i upon observing s_i are

$$l_i(a^{i-1}, \text{L}) = l_i(a^{i-1}) - l_Q, \qquad l_i(a^{i-1}, \ell) = l_i(a^{i-1}) - l_q,$$
$$l_i(a^{i-1}, \text{R}) = l_i(a^{i-1}) + l_Q, \qquad l_i(a^{i-1}, r) = l_i(a^{i-1}) + l_q.$$

The expected utility of player i with log likelihood ratio l from action $a_i = R$ if fraction f of previous players chose R is $\frac{\exp(l)}{1+\exp(l)} - fk$, and the expected utility from $a_i = L$ is $\frac{1}{1+\exp(l)} - (1-f)k$. The payoff difference $\Delta(l, f) := \frac{\exp(l)-1}{1+\exp(l)} + (1-2f)k$ determines the best response: player i chooses R if $\Delta(l, f) > 0$ and only if $\Delta(l, f) \geq 0$. Define the cutoff log likelihood ratio

$$l_k(f) := \ln(1 - k + 2fk) - \ln(1 + k - 2fk) \tag{1}$$

at which a player switches from action L to R. Clearly $l_k(\frac{1}{2}) = 0$ and $l_k(1) = -l_k(0)$ and more generally, $l_k(f) = -l_k(\frac{1}{2} - f)$ for any $f \geq \frac{1}{2}$.

The next section derives the optimal action choices of the players and provides sufficient conditions for herding to increase when players want to take a different action from their predecessors.

2 Beliefs and best responses

Player 1 chooses $a_1 = L$ after signals L, ℓ and $a_1 = R$ after R, r, due to the assumption $l_q > l_0$. There are no predecessors for player 1, so his optimal action does not depend on k. Similarly, if exactly half the predecessors of an odd-numbered player $2i - 1$ choose action L, then k does not affect a_{2i-1}.

Given $l_q > l_0$, player 2's log likelihood ratios conditional on player 1's action a_1 are $l_2(L) = l_0 - l_{Qq}$ and $l_2(R) = l_0 + l_{Qq}$ before observing s_2. The interpretation of $a_1 = L$ from player 2's perspective is as the 'average' of the signals L and ℓ, and similarly for $a_1 = R$. If the congestion cost is small and l_0 close to 0, then player 2 does not herd (a_2 responds to s_2). Lemma 1 characterises when the actions of the first two agents are informative.

Lemma 1. *Player 1's action is informative if $l_Q > l_0$ and only if $l_Q \geq l_0$. Player 2's action is informative after any a_1 if $l_q > l_0$, $l_0 - l_{Qq} - l_Q < l_k(0)$ and $l_0 + l_{Qq} - l_Q < l_k(1)$.*

Proof. If $l_Q > l_0$, then $l_1(\text{L}) < 0$, so $a_1(\text{L}) = L$. Due to $l_0 \geq 0$, $a_1(r) = a_1(\text{R}) = R$, thus a_1 is informative. If $l_Q < l_0$, then $l_1(\text{L}) > 0$, so $a_1 = R$ for any s_1.

Clearly $a_2(L, \text{R}) = R$ for any $k \geq 0$. Before observing s_2, if $l_q > l_0$, then $l_2(L) = l_0 - l_{Qq}$ and $l_2(R) = l_0 + l_{Qq}$. Then $l_0 - l_{Qq} - l_Q < l_k(0)$ implies $\Delta(l_2(L, \text{L}), f) < 0$, so $a_2(L, \text{L}) = L$, ensuring that a_2 is informative after $a_1 = L$.

The condition $l_0 + l_{Qq} + l_Q > l_k(1)$ ensuring $a_2(R, \text{R}) = R$ is implied by $l_0 - l_{Qq} - l_Q < l_k(0)$ and $l_k(0) = -l_k(1)$. If $l_0 + l_{Qq} - l_Q < l_k(1)$, then $a_2(R, \text{L}) = L$, thus a_2 is informative after $a_1 = R$. □

The maintained assumption $l_q > l_0$ implies $l_Q > l_0$, which ensures a_1 is informative by Lemma 1. The conditions sufficient for a_2 to be informative are not necessary. The interpretation of $l_0 - l_{Qq} - l_Q + l_k(1) < 0$ is that the congestion cost is small enough for player 2 not to ignore own signal just to take a different action from player 1. If $l_0 + l_{Qq} - l_Q - l_k(1) < 0$, then the prior probability of state $\mathcal{R}$ is low enough that a strong signal $s_2 = \mathrm{L}$ in favour of $\mathcal{L}$ together with the preference to differ outweighs the prior and player 1's action $a_1 = R$.

Next, sufficient conditions are provided for herding to increase after the introduction of the desire to differ from previous agents. Increased herding means that actions become uninformative after some histories, but not the reverse. The set of histories after which herding occurs under $k > 0$, but not under $k = 0$ can have probability close to 1, as the numerical example after Proposition 2 demonstrates. Proposition 2 proves increased herding for the first four players under $k > 0$ compared to $k = 0$. After that, Lemma 3 shows that player 5 also herds more under $k > 0$.

Proposition 2. *Assume $l_q > l_0$ and $l_0 - l_{Qq} - l_Q + l_k(1) < 0$.*
(a) If $k = 0$, $l_0 - l_{Qq} + l_q < 0$ and $l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$, then a_3 is informative after any a^2.
(b) If $k > 0$, $l_0 + l_{Qq} - l_q - l_k(1) < 0$ and $l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$, then a_3 is uninformative after $a_1 = a_2$, the probability of which is $(Q+q)^2 + (1-Q-q)^2 > \frac{1}{2}$. If in addition $l_0 + l_{Qq} - l_q - l_k\left(\frac{i+1}{2i+1}\right) < 0$, then a_{2i+3} is uninformative after $a_{2i+1} = a_{2i+2}$.
(c) If further $l_0 - l_{Qq} + l_{\neg Q} < 0$ and $a_4(a^3, s_3)$ is informative under $k > 0$, then also under $k = 0$.

Proof. (a) The condition $l_0 + l_{Qq} - l_Q - l_k(1) < 0$ is implied by $l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$ and by $l_0 + l_{Qq} - l_q - l_k(1) < 0$. Recall $l_k(f) = -l_k(\frac{1}{2} - f)$.

If $l_q > l_0$, then $a_1(\mathrm{L}) = a_1(\ell) = L$ and $a_1(\mathrm{R}) = a_1(r) = R$. In this case, $l_0 - l_{Qq} - l_Q + l_k(1) < 0$ ensures $a_2(L, \mathrm{L}) = L$. If $k = 0$, $l_0 - l_{Qq} + l_q < 0$ and $l_0 + l_{Qq} - l_Q - l_k(1) < 0$, then player 2's actions are $a_2(R, s_2 \neq \mathrm{L}) = R = a_2(L, \mathrm{R})$ and $a_2(R, \mathrm{L}) = L = a_2(L, s_2 \neq \mathrm{R})$, so

$$l_3(L, L) = l_0 - l_{Qq} - l_{\neg Q}, \qquad l_3(L, R) = l_0 - l_{Qq} + l_Q,$$
$$l_3(R, L) = l_0 + l_{Qq} - l_Q, \qquad l_3(R, R) = l_0 + l_{Qq} + l_{\neg Q}.$$

When $k = 0$ and $l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$, player 3's action is informative after $a_1 = a_2$:

private history (a^2, s_3)	$l_3(a^2, s_3)$	$a_3(a^2, s_3)$
R, R, L	$l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$	L
$R, R, s_3 \neq \mathrm{L}$	$\geq l_0 + l_{Qq} + l_{\neg Q} - l_q > 0$	R
L, L, R	$l_0 - l_{Qq} - l_{\neg Q} + l_Q > 0$	R
$L, L, s_3 \in \{\ell, \mathrm{L}\}$	$\leq l_0 - l_{Qq} - l_{\neg Q} - l_q < 0$	L

If $k = 0$, then a_3 is informative after $a_1 \neq a_2 = R$, because $l_3(L, R, \text{L}) = l_0 - l_{Qq} < 0$ due to $l_{Qq} > l_q > l_0$, and $l_3(L, R, r) = l_0 - l_{Qq} + l_Q + l_q > 0$. Action a_3 is informative after $a_1 \neq a_2 = L$, because $l_3(R, L, \text{R}) = l_0 + l_{Qq} > 0$ and $l_3(R, L, \text{L}) = l_0 + l_{Qq} - 2l_Q < 0$ due to $l_Q > l_{Qq} > l_0$.

(b) If $k > 0$, $l_q > l_0$, $l_0 + l_{Qq} - l_q - l_k(1) < 0$ and $l_0 - l_{Qq} + l_q + l_k(1) < 0$, then $a_2(a_1, \text{L}) = a_2(a_1, \ell) = L$ and $a_2(a_1, \text{R}) = a_2(a_1, r) = R$ for any a_1, so $l_3(L, L) = l_0 - 2l_{Qq}$, $l_3(L, R) = l_3(R, L) = l_0$ and $l_3(R, R) = l_0 + 2l_{Qq}$.

If $k > 0$ and $l_3(L, L, \text{R}) = l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$, then $l_3(L, L, s_3) < 0$ for any s_3, so $a_3(L, L, s_3) = L$ for any s_3. The condition $l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$ implies $l_0 + 2l_{Qq} - l_Q - l_k(1) > 0$, so $l_3(R, R, s_3) > 0$ and $a_3(R, R, s_3) = R$ for any s_3. If player 3 herds after $a_2 = a_1$, then so do all subsequent players, because $f = 1$ remains unchanged. More generally, if player i herds after some history a^{i-1} and $|f - \frac{1}{2}|$ weakly decreases over time, then all subsequent players $j > i$ also herd after any continuation of a^{i-1}.

In the $k > 0$ case, if $a_1 \neq a_2$, $l_0 + l_{Qq} - l_q - l_k(1) < 0$ and $l_0 - l_{Qq} + l_q + l_k(1) < 0$, then $l_3(a^2) = l_0$ and $f = \frac{1}{2}$, so a_3 is informative by $l_q > l_0$. For any period j, if $f = \frac{1}{2}$, then j is odd, $l_j(a^{j-1}) = l_0$ and if $a_{j+1} = a_j$, then in period $j + 2$, $f = \frac{(j-1)/2+2}{j+1}$. If $l_0 - l_{Qq} + l_q + l_k(1) < 0$, then $l_0 - l_{Qq} + l_q + l_k(f) < 0$ for any $f \leq 1$. Whenever $f = \frac{1}{2}$ and $l_j(a^{j-1}) = l_0$, the game essentially restarts, with player j in the role of player 1 and a reduced $l_k(f)$, because f responds less to $a_{j+1} = a_j$. Therefore if a herd has not started after a^{2i} (which implies $a_{2t} \neq a_{2t-1}$ for all $t \leq i$), then it starts after (a^{2i}, L, L), and if $l_0 + l_{Qq} - l_q - l_k\left(\frac{i+1}{2i+1}\right) < 0$, then also after (a^{2i}, R, R). The conditional probability of a herd is $\Pr(a_{2i+2} = a_{2i+1} | a_{2i+1} \neq a_{2i}) = 1 - 2(Q + q)(1 - Q - q) = (Q + q)^2 + (1 - Q - q)^2$.

(c) Table 1 displays $l_4(a^3)$ in the cases $k = 0$ and $k > 0$, as well as the conditions for $a_4(a^3, s_3)$ to be informative. Sufficient for $a_4(a^3, s_3)$ to be informative under $k = 0$ is that $l_4(a^3, \text{L}) < 0 < l_4(a^3, \text{R})$, which is how the fourth column of Table 1 is derived from the second. Under $k > 0$, if $a_1 = a_2$, then herding already started from a_3, so a_4 is uninformative. If $a_1 \neq a_2$, then player 4 faces the same decision problem as player 2, but with a smaller $|l_k(f)|$, so by Lemma 1, a_4 is informative for any a_3. More generally, if $a_{2t-1} \neq a_{2t}$ for all $t < i$, then player $2i$ faces the same decision problem as player 2, so by Lemma 1, a_{2i} is informative for any a_{2i-1}. Table 1 shows that if a_4 is informative under $k > 0$ and $l_0 - l_{Qq} + l_{\neg Q} < 0$, then a_4 is informative under $k = 0$. □

Proposition 2 is not vacuous—numerical examples satisfying the assumptions are presented next.

Example 1. Take $p_0 \in [0.5, 0.51]$, $p_S \approx 0.9$, $Q = 0.8991$, $q \approx 0.091$ and $k = 0.8173$. Alterna-

Table 1: Public log likelihood ratios of player 4 before seeing s_4, and conditions under which a_4 responds to s_4. Maintained assumptions: $l_0 - l_{Qq} + l_q < 0$, $l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$, $l_0 + l_{Qq} - l_q - l_k(1) < 0$ and $l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$.

	$l_4(a^3)$		$a_4(a^3, s_4)$ responds to s_4 if	
history a^3	$k = 0$	$k > 0$	$k = 0$	$k > 0$
L, L, L	$l_0 - l_{Qq} - 2l_{\neg Q}$	$l_0 - 2l_{Qq}$	$l_0 - l_{Qq} - 2l_{\neg Q} + l_Q > 0$	never
R, R, R	$l_0 + l_{Qq} + 2l_{\neg Q}$	$l_0 + 2l_{Qq}$	$l_0 + l_{Qq} + 2l_{\neg Q} - l_Q < 0$	never
L, L, R	$l_0 - l_{Qq} - l_{\neg Q} + l_Q$	off-path	always	never
R, R, L	$l_0 + l_{Qq} + l_{\neg Q} - l_Q$	off-path	always	never
L, R, L	$l_0 - l_{Qq} + l_Q - l_Q$	$l_0 - l_{Qq}$	always	always
R, L, R	$l_0 + l_{Qq} - l_Q + l_Q$	$l_0 + l_{Qq}$	always	always
L, R, R	$l_0 - l_{Qq} + l_Q + l_{\neg Q}$	$l_0 + l_{Qq}$	$l_0 - l_{Qq} + l_{\neg Q} < 0$	always
R, L, L	$l_0 + l_{Qq} - l_Q - l_{\neg Q}$	$l_0 - l_{Qq}$	always	always

tively, take $p_0 = \frac{5}{8}$, $p_S \approx 0.8$, $Q \approx 0.7991$, $q \approx 0.1917$ and $k \approx 0.773$. In both cases, player 3's herding probability increases from 0 to $(Q+q)^2 + (1-Q-q)^2 \approx 0.98$. Herding by player 4 (and 5, as Lemma 3 below shows) increases after every history.

The intuition for the condition $l_q > l_0$ in Proposition 2 is that player 1's weak signal outweighs the prior, so player 1 always follows own signal. The assumption $l_0 - l_{Qq} + l_q < 0$ ensures that with $k = 0$, the prior p_0 is close enough to $\frac{1}{2}$ for player 2's weak signal in favour of state $\mathcal{R}$ not to outweigh the "average" signal (which is player 1's action) favouring state $\mathcal{L}$. The best response of player 2 is then to follow player 1 except when s_2 is strong and disagrees with a_1.

From player 3's perspective, observing $a_2 \neq a_1$ under $k = 0$ is equivalent to seeing a strong signal $s_2 = a_2$, but observing $a_2 = a_1$ conflates the three other signals ℓ, r and $s_3 \in \{\text{L}, \text{R}\} \setminus \{a_2\}$, in which case 3's log likelihood ratio moves by only $l_{\neg Q}$. The intuition for $l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$ is that the effect l_Q of a strong signal outweighs the combined prior l_0, average signal l_{Qq} and the conflation $l_{\neg Q}$ of three signals when $k = 0$. Thus player 3 always follows a strong signal $s_3 \in \{\text{L}, \text{R}\}$, regardless of whether $s_3 \neq a_2$, so a_3 is informative.

Under $k > 0$, the assumption $l_0 + l_{Qq} - l_q - l_k(1) < 0$ ensures that player 2 always follows own signal, because the desire to differ[1] from player 1 combines with the effect of a weak signal to outweigh the prior and the information derived from a_1. This choice of player 2 to

[1] If $l_0 - l_{Qq} - l_q < -l_k(1)$, which is implied by $l_0 - 2l_{Qq} + l_Q < -l_k(1)$ and $l_Q > l_{Qq}$, then player 2 does not ignore s_2 just to ensure $a_2 \neq a_1$, i.e. k is small enough not to induce anti-herding.

follow s_2 increases the informativeness of $a_2 = a_1$, but decreases that of $a_2 \neq a_1$. The more informative event $a_2 = a_1$ together with $l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$ induces player 3 to herd, because even a strong signal plus the desire to differ $l_k(1)$ do not overcome the effect $2l_{Qq}$ of two “average” signals. If player 3 herds after $a_2 = a_1$, then so do all subsequent players, because they have the same signal strengths and desire to differ.

The less informative $a_2 \neq a_1$ under $k > 0$ does not reduce player 3’s herding, because even under $k = 0$, player 3 follows a strong signal after $a_2 \neq a_1$. No additional assumptions are needed, because $a_2 \neq a_1$ is either a strong signal (if $k = 0$) or average (if $k > 0$) favouring the opposite state to a_1. The average signal from $a_2 \neq a_1$ neutralises a_1, so $l_q > l_0$ is sufficient for a_3 to respond to even weak signals. The strong signal from $a_2 \neq a_1$ under $k = 0$ is neutralised by player 3’s strong private signal $s_3 \in \{\mathrm{L}, \mathrm{R}\} \setminus \{a_2\}$, in which case $a_3 = a_1$. On the other hand, if $s_3 = a_2$, then $a_3 = a_2 \neq a_1$, so the action of player 3 is informative in the $k = 0$ case as well.

In Table 1, the condition $l_0 + l_{Qq} + 2l_{\neg Q} - l_Q < 0$ on the line R, R, R is sufficient for $l_0 - l_{Qq} - 2l_{\neg Q} + l_Q > 0$ on the L, L, L line. The intuition for these conditions is that a strong signal s_4 overwhelms the effect of an “average” signal from a_1 plus two conflations (a_2 and a_3) of the three signals other than a strong one opposing a_1. The condition $l_0 + l_{Qq} - l_{\neg Q} > 0$ for $a_4(R, L, L, s_4)$ to respond to s_4 under $k = 0$ always holds (so is omitted from the last line of Table 1), because $l_0 \geq 0$ and $l_{Qq} > l_{\neg Q}$. The maintained assumption $l_0 - l_{Qq} + l_q < 0$ is logically independent of the condition $l_0 - l_{Qq} + l_{\neg Q} < 0$ ensuring an informative $a_4(L, R, R, s_4)$ (penultimate line in Table 1), because both $l_q > l_{\neg Q}$ and $l_q < l_{\neg Q}$ are possible. The reason why $l_0 - l_{Qq} + l_{\neg Q} < 0$ is sufficient for $a_4(L, R, R, s_4)$ to respond to s_4 is that the strong signal from $a_2 \neq a_1 = L$ is cancelled by $s_4 = \mathrm{L}$, resulting in $l_4(L, R, R, \mathrm{L}) < 0$, but if $s_4 \in \{r, \mathrm{R}\}$, then $l_4(L, R, R, s_4) > 0$.

The next lemma compares the informativeness of a_5 under $k = 0$ to the $k > 0$ case. It complements Proposition 2 by showing that in addition to the increased herding by the first four agents, player 5 also responds less to signals. A similar result for player 6 is subsequently derived in Lemma 4.

Lemma 3. *If $l_0 - l_{Qq} + l_{\neg Q} < 0$ and the assumptions of Proposition 2 hold and $a_5(a^4, s_5)$ is informative under $k > 0$, then $a_5(a^4, s_5)$ is also informative under $k = 0$.*

Proof. History $a^3 = (R, L, L)$. Under $k = 0$, if $l_4(R, L, L, r) = l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_q < 0$, then player 5’s log likelihood ratios continuing from $a^3 = (R, L, L)$ are $l_5(R, L, L, L) = l_0 + l_{Qq} - l_Q - l_{\neg Q} - l_{\neg Q}$ and $l_5(R, L, L, R) = l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_Q$, because player 4 chooses L after a weak signal $s_4 = r$. In this case, a_5 responds to s_5 if $l_0 + l_{Qq} - 2l_{\neg Q} > 0$, because

$l_5(R, L, L, L, \text{R}) = l_0 + l_{Qq} - 2l_{\neg Q}$ and $l_5(R, L, L, L, \text{L}) = l_0 + l_{Qq} - 2l_Q - 2l_{\neg Q} < 0$. After (R, L, L, R), player 5's action always responds to the private signal. By comparison, recall that when $k > 0$, player 5 (and any odd player) herds if the preceding two players took the same action.

On the other hand, if $l_4(R, L, L, r) > 0$, then $l_5(R, L, L, L) = l_0 + l_{Qq} - l_Q - l_{\neg Q} - l_{Qq}$ and $l_5(R, L, L, R) = l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_{Qq}$, because player 4 chooses L after a weak signal $s_4 = r$. In this case, a_5 responds to s_5 if $l_5(R, L, L, L, \text{R}) = l_0 - l_{\neg Q} > 0$ (again, a_5 always responds if the a^4 contains an equal number of L, R). The condition $l_0 - l_{\neg Q} > 0$ fails in Example 1 above, so player 5 herds. When $k > 0$, player 5 always herds after history (R, L, L, L).

History $a^3 = (L, R, R)$. If $l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q > 0$, then $l_5(L, R, R, L) = l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_Q$ and $l_5(L, R, R, R) = l_0 - l_{Qq} + l_Q + l_{\neg Q} + l_{\neg Q}$, because $a_4(L, R, R, \ell) = R$. The condition for a_5 to respond to s_5 is $l_5(L, R, R, R, \text{L}) = l_0 - l_{Qq} + l_Q + l_{\neg Q} + l_{\neg Q} - l_Q < 0$, the same as for a_4 to be informative after $a^3 = (L, R, R)$.

In contrast, if $l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q < 0$, then $l_5(L, R, R, L) = l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_{Qq}$ and $l_5(L, R, R, R) = l_0 - l_{Qq} + l_Q + l_{\neg Q} + l_{Qq}$. Action a_5 is always informative after L, R, R, L, but never after L, R, R, R (just like with $k > 0$), because $l_5(L, R, R, R, \text{L}) = l_0 + l_{\neg Q} > 0$.

Histories $a^3 = (R, L, R)$ and (L, R, L) lead to $l_4(a^3) = l_0 \pm l_{Qq}$, so player 4 faces the same decision problem as player 2. Thus continuing from these histories, any player herds more under $k > 0$ than under $k = 0$. For histories in the top half of Table 1, herding has already started with player 3, so all subsequent players unambiguously herd more under $k > 0$. □

Lemma 4 compares the informativeness of the action a_6 of player 6 under $k = 0$ and $k > 0$, analogously to Lemma 3 for a_5.

Lemma 4. *If $l_0 + 2l_{Qq} - l_{\neg Q} - l_Q < 0$ and $a_6(a^5, s_6)$ is informative under $k > 0$, then $a_6(a^5, s_6)$ is also informative under $k = 0$.*

Proof. Based on Proposition 2 and Lemma 3, the only histories continuing from which player 6 could conceivably herd more under $k = 0$ are $a^4 = (L, R, R, L)$ and (R, L, L, R). In these continuations, under $k > 0$, player 6 faces the same decision as player 2, but this need not be the case under $k = 0$. Consider first $a^4 = (R, L, L, R)$. Separate two cases based on the sign of $l_4(R, L, L, r) = l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_q$.

If $l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_q < 0$, then $l_5(R, L, L, R) = l_0 + l_{Qq} - l_{\neg Q}$. In this case, if $l_0 + l_{Qq} - l_{\neg Q} - l_q < 0$, then $l_6(R, L, L, R, L) = l_0 - l_{\neg Q}$ (so a_6 is informative) and $l_6(R, L, L, R, R) = l_0 + 2l_{Qq} - l_{\neg Q}$. Then a_6 is informative if $l_0 + 2l_{Qq} - l_{\neg Q} - l_Q < 0$.

The other case given $l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_q < 0$ is $l_0 + l_{Qq} - l_{\neg Q} - l_q > 0$, which implies $l_6(R, L, L, R, L) = l_0 + l_{Qq} - l_{\neg Q} - l_Q$ and $l_6(R, L, L, R, R) = l_0 + l_{Qq} - l_{\neg Q} + l_{\neg Q}$, for both of which, a_6 is informative.

If $l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_q > 0$, then $l_5(R, L, L, R) = l_0 + 2l_{Qq} - l_Q - l_{\neg Q}$. In this case, if $l_0+2l_{Qq}-l_Q-l_{\neg Q}-l_q < 0$ (implied by $l_0+2l_{Qq}-l_Q-l_{\neg Q} < 0$) and $l_0+2l_{Qq}-l_Q-l_{\neg Q}+l_q > 0$ (implied by $l_0 + l_{Qq} - l_Q - l_{\neg Q} + l_q > 0$), then $l_6(R, L, L, R, L) = l_0 + l_{Qq} - l_Q - l_{\neg Q}$ (so a_6 is informative) and $l_6(R, L, L, R, R) = l_0 + 3l_{Qq} - l_Q - l_{\neg Q}$. Therefore if $l_0 + 3l_{Qq} - 2l_Q - l_{\neg Q} < 0$ (which is implied by $l_0 + 2l_{Qq} - l_Q - l_{\neg Q} < 0$), then a_6 is informative.

Consider next $a^4 = (L, R, R, L)$, so $l_4(L, R, R, \ell) = l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q$. If $l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q > 0$, then $l_5(L, R, R, L) = l_0 - l_{Qq} + l_{\neg Q}$. In this case, if $l_0 - l_{Qq} + l_{\neg Q} + l_q > 0$, then $l_6(L, R, R, L, R) = l_0 + l_{\neg Q}$ (so a_6 is informative) and $l_6(L, R, R, L, L) = l_0 - 2l_{Qq} + l_{\neg Q}$. Then a_6 is informative if $l_0 - 2l_{Qq} + l_{\neg Q} + l_Q > 0$, sufficient for which is $l_0 + 2l_{Qq} - l_{\neg Q} - l_Q < 0$.

The other case given $l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q > 0$ is $l_0 - l_{Qq} + l_{\neg Q} + l_q < 0$, which implies $l_6(L, R, R, L, R) = l_0 - l_{Qq} + l_{\neg Q} + l_Q$ and $l_6(L, R, R, L, L) = l_0 - l_{Qq} + l_{\neg Q} - l_{\neg Q}$, so $a_6(L, R, R, L, L, s_6)$ is informative. Action $a_6(L, R, R, L, R, s_6)$ is informative if $l_0-l_{Qq}+l_{\neg Q} < 0$, which is implied by $l_0 - l_{Qq} + l_{\neg Q} + l_q < 0$.

If $l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q < 0$, then $l_5(L, R, R, L) = l_0 - 2l_{Qq} + l_Q + l_{\neg Q}$. In this case, if $l_0 - 2l_{Qq} + l_Q + l_{\neg Q} + l_q > 0$ (which is implied by $l_0 + 2l_{Qq} - l_{\neg Q} - l_Q < 0$) and $l_0 - 2l_{Qq} + l_Q + l_{\neg Q} - l_q < 0$ (implied by $l_0 - l_{Qq} + l_Q + l_{\neg Q} - l_q < 0$), then $l_6(L, R, R, L, R) = l_0-l_{Qq}+l_Q+l_{\neg Q}$, so $a_6(L, R, R, L, R, s_6)$ is informative, because $l_0-l_{Qq}+l_Q+l_{\neg Q}-l_q < 0$ implies $l_0-l_{Qq}+l_{\neg Q} < 0$. Also, $l_6(L, R, R, L, L) = l_0-3l_{Qq}+l_Q+l_{\neg Q}$, thus if $l_0-3l_{Qq}+2l_Q+l_{\neg Q} > 0$ (sufficient for which is $l_0 + 2l_{Qq} - l_{\neg Q} - l_Q < 0$), then a_6 is informative. □

The assumption $l_0 + 2l_{Qq} - l_{\neg Q} - l_Q < 0$ in Lemma 4 that suffices for player 6 to herd more under $k > 0$ is satisfied in Example 1 above. Therefore the set of histories in which the first six players herd under $k > 0$ is a proper superset of the histories in which they herd under $k = 0$.

In some long enough histories, the probability of herding under $k = 0$ may overtake that under $k > 0$. This is because the effective congestion cost decreases when f approaches $\frac{1}{2}$, which occurs each time the history lengthens by two actions without a herd having started.

Eyster et al. (2014) show that as the congestion cost approaches zero, learning increases in the limit as time goes to infinity. The comparison of the limiting probabilities of learning under $k = 0$ and k bounded away from zero is complicated, because it depends on the speed of convergence of $|f - \frac{1}{2}|$.

3 Comparing $k = 0$ and $k > 0$ using a coupling argument

Why coupling may not work (a ‘coupled’ action pattern eventually ‘uncouples’):

A herd *breaking* is defined as actions becoming informative again after being uninformative for some past agent: $\exists i < j$ s.t. a_i is constant in s_i, but a_j is not.

If herding starts under $k > 0$ at log likelihood ratio $l_0 \pm 2l_{Qq}$ and some f, then the increase in $|f - \frac{1}{2}|$ due to agents taking the same action does not break the herd, because by the assumption $l_0 - 2l_{Qq} + l_Q < l_k(0)$, even $f \in \{0, 1\}$ does not break the herd.

On the other hand, if actions alternate long enough, then $l_k(f)$ becomes arbitrarily small. Eventually, agents start ignoring their weak signals $s_i \in \{\ell, r\}$, so the log likelihood ratio no longer equals $l_0 \pm l_{Qq}$ or l_0, even if a herd has not started. Instead, l_i enters the set of log likelihood ratios that occur under $k = 0$. This set contains $l_0 \pm (l_{Qq} + l_{\neg Q})$, $l_0 \pm (l_{Qq} - l_Q)$, $l_0 \pm (l_{Qq} - l_Q - l_{\neg Q})$, $l_0 \pm (l_{Qq} + 2l_{\neg Q})$, etc. If herding starts after the log likelihood ratio leaves $\{l_0, l_0 \pm l_{Qq}, l_0 \pm 2l_{Qq}\}$, then the herd may eventually break due to the increase in $|f - \frac{1}{2}|$. Whether it breaks depends on the f and l at which it starts. For example, if the herd starts at $l_0 + l_{Qq} + nl_{\neg Q} \geq l_0 + 2l_{Qq}$, then it never breaks, but if it starts at $l_0 + l_{Qq} + nl_{\neg Q} < l_k(1) + l_Q < l_0 + 2l_{Qq}$, then it eventually breaks. The starting log likelihood ratio has smaller absolute value when $|f - \frac{1}{2}|$ is smaller, in which case the herd is more likely to break.

If the actions alternate long enough, then $l_k(f) \approx 0$ and if a herd starts, then $|l_k(f)|$ eventually grows to $l_k(1)$. A herd starting with f does not break for example when $[l_0 + l_{Qq} + nl_{\neg Q} - l_Q \in (l_k(f), l_{\neg Q} + l_k(f))$ implies $l_0 + l_{Qq} + nl_{\neg Q} - l_Q > l_k(1)]$. The condition $l_0 + l_{Qq} + nl_{\neg Q} - l_Q < l_{\neg Q} + l_k(f)$ ensures the herd does not start at $l_0 + l_{Qq} + (n-1)l_{\neg Q}$. Breaking a herd that starts after a long enough alternation is guaranteed if $l_{\neg Q} < l_k(1)$.

Rewrite $l_0 + l_{Qq} + nl_{\neg Q} - l_Q \in (l_k(f), l_{\neg Q} + l_k(f)) \Rightarrow l_0 + l_{Qq} + nl_{\neg Q} - l_Q > l_k(1)$ as $n \in \left(\frac{l_k(f) - l_0 - l_{Qq} + l_Q}{l_{\neg Q}}, \frac{l_k(f) - l_0 - l_{Qq} + l_Q}{l_{\neg Q}} + 1\right) \Rightarrow n > \frac{l_k(1) - l_0 - l_{Qq} + l_Q}{l_{\neg Q}}$. This holds iff there is no integer $n \in \left(\frac{-l_0 - l_{Qq} + l_Q}{l_{\neg Q}}, \frac{l_k(1) - l_0 - l_{Qq} + l_Q}{l_{\neg Q}}\right)$.

The next proposition guarantees that after every history, actions are more informative under $k > 0$ than $k = 0$.

even-numbered agents are a problem after l under $k > 0$ leaves $l_0 \pm \{0, 1, 2\}\, l_{Qq}$, bec may herd under $k = 0$ and not $k > 0$, unless coupling takes care of it

Proposition 5. *For any i, a^{i-1}, if the assumptions in Proposition 2 ($l_q > l_0$, $l_0 - l_{Qq} - l_Q + l_k(1) < 0$, $l_0 - l_{Qq} + l_q < 0$, $l_0 + l_{Qq} + l_{\neg Q} - l_Q < 0$, $l_0 + l_{Qq} - l_q - l_k(1) < 0$, $l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$ and $l_0 - l_{Qq} + l_{\neg Q} < 0$) hold and *conditions* and $a_i(a^{i-1}, s_i)$ is constant in s_i under $k > 0$,* *then $a_i(a^{i-1}, s_i)$ is also constant in s_i under $k = 0$.*

Can bound below the $|l_k(f)|$ at which l leaves $l_0 \pm \{0,1,2\}\, l_{Qq}$, so *can alternate arb long before herding, so the bound is alw $l_k(1)$* can bound above the increase in $|l_k(f)|$ that occurs during herding under $k > 0$. Start ignoring the weak signal ℓ at $f_\ell > \frac{1}{2}$ s.t. $l_0 + l_{Qq} - l_q > l_k(f_\ell)$. Start ignoring r at $f_r < \frac{1}{2}$ s.t. $l_0 - l_{Qq} + l_q < l_k(f_r)$. If herding has not started and $l_{2(i+1)}(a^{2i+1}) = l_0 + l_{Qq}$, then $f = \frac{i+1}{2i+1}$ and if $l_{2(i+1)}(a^{2i+1}) = l_0 - l_{Qq}$, then $f = \frac{i}{2i+1}$. Other values of f become possible once l leaves the set $l_0 \pm \{0,1,2\}\, l_{Qq}$.

The necessary and sufficient conditions for herd-breaking are as follows.

Proposition 6. *If the assumptions of Prop. 2 hold, then*

1. *a herd that starts at $l_i = l_0 \pm 2l_{Qq}$ or under $k = 0$ does not break,*
2. **not necessarily* under $k > 0$, a herd that starts at $l_i \neq l_0 \pm 2l_{Qq}$ breaks eventually.*

Proof. Herding implies $l_{i+1} = l_i$. At $l_i \geq 0$, a herd starts iff $l_i - l_Q > l_k(f)$ and breaks if the reverse inequality holds. At $l_i < 0$, a herd starts iff $l_i + l_Q < l_k(f)$ and breaks if the reverse inequality holds. Under $k = 0$, the cutoff $l_k(f)$ stays constant at 0, so a herd that starts never breaks.

Under $k > 0$, the assumption $l_0 - 2l_{Qq} + l_Q + l_k(1) < 0$ in Prop. 2 ensures that a herd starting at $l_i = l_0 \pm 2l_{Qq}$ does not break at the maximal desire to differ $l_k(1)$.

incomplete Define

$$n_{++} := \min\{n : l_0 + l_{Qq} + nl_{\neg Q} - l_Q > l_k(1)\},$$
$$n_{--} := \min\{n : l_0 - l_{Qq} - nl_{\neg Q} + l_Q < l_k(0)\},$$
$$n_{-+} := \min\{n : l_0 - l_{Qq} + l_Q + nl_{\neg Q} > l_k(1)\},$$
$$n_{+-} := \min\{n : l_0 + l_{Qq} - l_Q - nl_{\neg Q} < l_k(0)\}$$

The public log likelihood ratio process $(l_i(a^{i-1}))_{i=1}^{\infty}$ only takes values l_0, $l_0 \pm l_{Qq}$, $l_0 \pm 2l_{Qq}$, $l_0 \pm (l_{Qq} + nl_{\neg Q})$ for $n = 1, \ldots, \max\{n_{++}, n_{--}\}$ and $l_0 \pm (l_{Qq} - l_Q - nl_{\neg Q})$ for $n = 1, \ldots, \max\{n_{+-}, n_{-+}\}$ (not necessarily all these values). This is because each time the process crosses zero from below, either l_{Qq} or l_Q is added and each time the process crosses zero from above, l_{Qq} or l_Q is subtracted. The added and subtracted terms cancel, so $l_i(a^{i-1})$ contains at most one each of $\pm l_{Qq}$ and $\pm l_Q$.

A herd that starts at $l_0 - l_{Qq} - nl_{\neg Q}$ for some n breaks eventually iff $l_k(0) < l_0 - l_{Qq} - nl_{\neg Q} + l_Q$, because l remains constant as long as the herd continues. A herd that starts at $l_0 + l_{Qq} + nl_{\neg Q}$ for some n breaks eventually iff $l_0 + l_{Qq} + nl_{\neg Q} - l_Q < l_k(1)$. A herd that starts at $l_0 + l_{Qq} - l_Q - nl_{\neg Q}$ for some n breaks eventually iff $l_k(0) < l_0 + l_{Qq} - nl_{\neg Q}$. A herd that starts at $l_0 - l_{Qq} + l_Q + nl_{\neg Q}$ for some n breaks eventually iff $l_0 - l_{Qq} + nl_{\neg Q} < l_k(1)$.

The assumption $l_0 + l_k(1) < 2l_{Qq} - l_Q$ in Prop. 2 implies $l_k(1) < l_{Qq} < l_Q$. As long as actions stay uninformative, f converges to either 0 or 1, so $l_k(f)$ converges to $l_k(0)$ or $l_k(1)$. A herd starts at $l_0 - l_{Qq} - nl_{\neg Q}$ if $l_0 - l_{Qq} - nl_{\neg Q} + l_Q < l_k(f)$, with $f \leq 0$.

maybe small difference betw $l_k(f)$ and $l_k(1)$ A herd starts at $l_0 - l_{Qq} - nl_{\neg Q}$ if $l_0 - l_{Qq} - (n-1)l_{\neg Q} + l_Q > l_k(f_{n-1})$ and $l_0 - l_{Qq} - nl_{\neg Q} + l_Q < l_k(f_n)$, with $f_{n-1}, f_n \leq 0$. □

To do: Define cancelling in terms of l and future incentives remaining unchanged. Markovian continuation game the same (not a subgame due to incomplete info).

Lemma 7. *Suppose the assumptions of Lemmas 1–3 and Proposition 2 hold. Blocks of two actions cancel under $k > 0$ before l leaves $l_0 \pm \{0, 1, 2\}\, l_{Qq}$, and in the same set of histories, blocks of four actions always cancel under $k = 0$.*

Proof. If $a_{2i} = a_{2i-1}$ for some $i \in \mathbb{N}$, and $|f - \frac{1}{2}|$ is large enough, then under $k > 0$, a herd starts. Although $|f - \frac{1}{2}|$ may increase as the herd continues, actions never become informative again, because $l_0 - 2l_{Qq} + l_Q < l_k(0)$. Removing blocks (a_{2i-1}, a_{2i}) from the public history after the herd has started leaves the continuation game unchanged.

Focus on $a_{2i} \neq a_{2i-1}\ \forall i \leq \frac{j-1}{2}$. Under $k > 0$, $l_j(a^{2i}) = l_0$. Suppose $a_{2i} = R$ (the proof for L is symmetric). Under $k = 0$, $l_j(a^{2i}) = l_{2i}(a^{2i-1}) + l_Q$ by the assumption $a_{2i} \neq a_{2i-1} = L$. If $a_{2i-1} \neq a_{2i-2}$, then $l_{2i}(a^{2i-1}) = l_{2i-1}(a^{2i-2}) - l_Q$, so $l_j(a^{2i}) = l_{2i-1}(a^{2i-2})$. Removing the two-action block (a_{2i-1}, a_{2i}) from the public history leaves the log likelihood ratio and future incentives unchanged—the changing f does not matter due to $k = 0$.

Focus on the unique history in which $a_{2i} \neq a_{2i-1} = a_{2i-2}$ for all $i \leq \frac{j-1}{2}$. If $a_{2i-1} = a_{2i-2} = L$, then $l_{2i}(a^{2i-1}) = l_{2i-1}(a^{2i-2}) - l_{\neg Q}$, so $l_j(a^{2i}) = l_{2i-1}(a^{2i-2}) + l_Q - l_{\neg Q}$. Applying the same reasoning to agent $2i - 1$ as to j, but switching L, R shows that $l_{2i-1}(a^{2i-2}) = l_{2i-3}(a^{2i-4}) - l_Q + l_{\neg Q}$. Therefore $l_j(a^{2i}) = l_{2i-3}(a^{2i-4})$. Removing the four-action block $(a_{2i-3}, \ldots, a_{2i})$ from the public history leaves the log likelihood ratio and future incentives unchanged. □

4 The probability of the correct action

This section calculates different players' probabilities of matching the true state with their action, with and without the desire to differ from previous movers. Under $k = 0$, each agent's probability of matching the state is greater than that of any previous agent, because later agents have better information: the same precision of the private signal and more observations of past actions. With $k > 0$, the desire to differ may reduce later agents' likelihood of the correct action.

Let the initial log likelihood ratio be $l_0 \in \mathbb{R}$. As established in the previous section, the actions of agents 2 to 5 are less informative under a desire to differ under the following **Assumptions**:

1. $|l_0| < l_q$,
2. $|l_0| - l_{Qq} - l_Q < l_k(0)$,
3. $|l_0| - l_{Qq} + l_q < 0$,
4. $|l_0| + l_{Qq} + l_{\neg Q} - l_Q < 0$,
5. $|l_0| + l_{Qq} - l_q < l_k(1)$,
6. $|l_0| - 2l_{Qq} + l_Q + l_k(1) < 0$,
7. $|l_0| - l_{Qq} + l_{\neg Q} < 0$.

The absolute value of l_0 is used to allow for $l_0 < 0$ in this section.

For the first agent, $\Pr(a_1 = \theta) = \Pr(a_1 = \mathcal{R}|\mathcal{R}) = Q + q$ regardless of k. For player 2, without a preference to mismatch the previous agent's action, $\Pr(a_2 = \theta|k = 0) = \Pr(a_2 = \mathcal{R}|k = 0, \mathcal{R}) = (Q + q)(1 - p_S + Q) + (1 - Q - q)Q = (Q + q)(1 - p_S) + Q$. With such a preference, $\Pr(a_2 = R|k > 0, \mathcal{R}) = Q + q$. Unsurprisingly, $Q + q < (Q + q)(1 - p_S) + Q$—the probability of matching the state is lower when an agent's goal is not just to match the state, but also to differentiate his action from those of previous agents. The difference of agent 2's behaviour under $k > 0$ from $k = 0$ arises when agent 2 receives a weak signal that suggests the opposite state to 1's action (e.g. ℓ after $a_1 = R$). Under $k > 0$, agent 2 follows such a signal, unlike under $k = 0$. If the signal is correct (agent 1's action did not equal the state), then having $k > 0$ improves the state-matching of a_2, otherwise worsens it. The probability of matching the state better is $(1 - Q - q)q$, and the probability of a worse match is $(Q + q)(1 - p_S - q)$. The latter exceeds $(1 - Q - q)q$ iff $Q(1 - p_S) > p_S q$, which is the definition of ℓ, r being weaker signals than L, R.

Without a desire to differ, $\Pr(a_3 = R|\mathcal{R}) = (Q + q)[(1 - p_S + Q)^2 + (p_S - Q)Q] + (1 - Q - q)[Q(1 - p_S + Q) + (1 - Q)Q]$. When agents want to mismatch previous actions, $\Pr(a_3 = R|k > 0, \mathcal{R}) = (Q + q)^2 + 2(Q + q)(1 - Q - q)(Q + q)$. Parameters at which the probability of matching the state is greater under a desire to differ are those in Example 1: either $p_0 \in [0.5, 0.51]$, $p_S \approx 0.9$, $Q = 0.8991$, $q \approx 0.091$ and $k = 0.8173$, or $p_0 = \frac{5}{8}$, $p_S \approx 0.8$, $Q \approx 0.7991$, $q \approx 0.1917$ and $k \approx 0.773$. If $p_0 = \frac{1}{2}$, then a_3 more closely matches the state under a desire to differ for all values of the other parameters, according to Mathematica. Surprisingly, agent 3 matches the state more when this is not the sole goal of all agents. Further, a_3's closer tracking of the state occurs despite 3 imitating 2 more and a_2 diverging from the state with a greater probability.

With $k = 0$, the probability of 4 matching the state is $\Pr(a_4 = R|\mathcal{R})$
$= (1-Q-q)(1-Q)^2 Q \mathbf{1}\{l_0 - l_{Qq} - 2l_{\neg Q} + l_Q > 0\}$
$+ (1-Q-q)(1-Q)Q[Q + q + (1-p_S-q)\mathbf{1}\{l_0 - l_{Qq} - l_{\neg Q} + l_Q - l_q > 0\}]$
$+(1-Q-q)Q[(p_S-Q)Q+(1-p_S-q)\mathbf{1}\{l_0 - l_{Qq} + l_Q - l_q < 0\}(Q+q\mathbf{1}\{l_0 - 2l_{Qq} + l_Q + l_q > 0\})]$
$+ (1-Q-q)Q[Q + q + (1-p_S-q)\mathbf{1}\{l_0 - l_{Qq} + l_Q - l_q > 0\}]^2$
$+ (Q+q)(p_S-Q)[(1-Q-q)Q + q\mathbf{1}\{l_0 + l_{Qq} - l_Q + l_q < 0\} Q]$
$+(Q+q)(p_S-Q)[Q(1-p_S+Q)+q\mathbf{1}\{l_0 + l_{Qq} - l_Q + l_q > 0\}[Q+q+(1-p_S-q)\mathbf{1}\{l_0 + 2l_{Qq} - l_Q - l_q > 0\}]]$
$+ (Q+q)(1-p_S+Q)(p_S-Q)(Q + q\mathbf{1}\{l_0 + l_{Qq} + l_{\neg Q} - l_Q + l_q > 0\})$
$+(Q+q)(1-p_S+Q)^2[p_S - Q + (1-p_S+Q)\mathbf{1}\{l_0 + l_{Qq} + 2l_{\neg Q} - l_Q > 0\}]$. Under a preference to differ, $\Pr(a_4 = R|k > 0, \mathcal{R}) = (Q+q)^2(3 - 2Q - 2q) = \Pr(a_3 = R|k > 0, \mathcal{R})$. If $p_0 = \frac{1}{2}$, then Mathematica shows that for all values of the other parameters, agent 4 matches the state better under $k > 0$.

The higher probability of a_3 and a_4 matching the state under $k > 0$ resembles the improved learning in the limit (as the number of agents who have moved grows) in Eyster et al. (2014) under a small positive k compared to $k = 0$.

5 General signal distribution

In this section, each agent i draws a private signal $s_i \in [s_{\min}, s_{\max}] \subset \mathbb{R}$ from the cdf $F_s(\cdot|\theta)$, i.i.d. conditional on the state θ. Discrete and continuous signal structures are covered by this framework. Signals are assumed imperfectly informative, i.e. $F_s(\cdot|\mathcal{L}), F_s(\cdot|\mathcal{R})$ have common support and are mutually absolutely continuous, in particular any mass points of $F_s(\cdot|\mathcal{L})$ and $F_s(\cdot|\mathcal{R})$ coincide. Denote by $dF_s(x|\theta)$ the density of $F_s(\cdot|\theta)$ at x if it exists, otherwise the height of the jump of $F_s(\cdot|\theta)$ at x. Assume the monotone likelihood ratio property (MLRP): $\frac{dF_s(x|\mathcal{R})}{dF_s(x|\mathcal{L})}$ strictly increases in x, which implies (a) $F_s(x|\mathcal{R}) < F_s(x|\mathcal{L})$ for all $x \in (s_{\min}, s_{\max})$, (b) $\frac{F_s(x|\mathcal{R})}{F_s(x|\mathcal{L})}$ strictly increases in x and (c) $\frac{1-F_s(x|\mathcal{R})}{1-F_s(x|\mathcal{L})}$ strictly decreases.

W.l.o.g. label the signals so that $\ln \frac{dF_s(s_i|\mathcal{R})}{dF_s(s_i|\mathcal{L})} = s_i$, i.e. signal s shifts the log likelihood ratio by s. This labelling is somewhat similar to Smith and Sørensen (2000) who endow each agent with a private likelihood ratio incorporating the prior. The posteriors aggregate to the prior, so $s_{\min} < 0 < s_{\max}$, $dF_s(z|\mathcal{R}) = \exp(z)dF_s(z|\mathcal{L})$ and $\int_{s_{\min}}^{s_{\max}} \exp(z)dF_s(z|\mathcal{L}) = 1$.

Agent 1 optimally chooses L if $l_0 + \ln \frac{dF_s(s_1|\mathcal{R})}{dF_s(s_1|\mathcal{L})} < 0$ and chooses R if $l_0 + s_1 \geq 0$. To make agent 1's decision nontrivial, assume $s_{\min} < -l_0$. Due to $l_0 \geq 0$, $s_{\max} > 0$ implies $-l_0 < s_{\max}$.

For agent 2, the log likelihood ratios before observing s_2 are $l_2(R) = l_0 + \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}$ if 1 chose R, and $l_2(L) = l_0 + \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}$ if $a_1 = L$. The first two agents' log likelihood ratios are independent of k.

Agent i's best response is L if $l_i(a^{i-1}) + \ln \frac{dF_s(s_i|\mathcal{R})}{dF_s(s_i|\mathcal{L})} < l_k(f)$ and R if $s_i \geq l_k(f) - l_i(a^{i-1})$. Therefore agent i herds after a^{i-1} if either $s_{\min} \geq l_k(f) - l_i(a^{i-1})$ or $s_{\max} < l_k(f) - l_i(a^{i-1})$. Conditional on history $a^{i-1} = (a^{i-2}, a_{i-1})$ and cost k, agent i's log likelihood ratio $l_i^k(a^{i-1})$ before seeing s_i is calculated iteratively by

$$l_i^k((a^{i-2}, L)) = l_{i-1}^k(a^{i-2}) + \ln \frac{F_s\left(l_k(f_{-2}) - l_{i-1}^k(a^{i-2})|\mathcal{R}\right)}{F_s\left(l_k(f_{-2}) - l_{i-1}^k(a^{i-2})|\mathcal{L}\right)},$$
$$l_i^k((a^{i-2}, R)) = l_{i-1}^k(a^{i-2}) + \ln \frac{1 - F_s\left(l_k(f_{-2}) - l_{i-1}^k(a^{i-2})|\mathcal{R}\right)}{1 - F_s\left(l_k(f_{-2}) - l_{i-1}^k(a^{i-2})|\mathcal{L}\right)},$$

where f_{-2} is the fraction of actions R in a^{i-2}. Clearly $l_i^k(a^{i-1})$ is a Markov process, with state variables f_{-2} and $l_{i-1}^k(a^{i-2})$, and if $k = 0$, then just $l_{i-1}^k(a^{i-2})$.

The probability that a_2 matches a_1 falls in k, but a similar result need not hold for subsequent agents, because $l_i^k(a^{i-1})$ varies in k. Only conditional on $l_i^k(a^{i-1})$ does the probability of a_i matching the majority of previous agents' actions decrease in $|l_k(f)|$.

Proposition 8. *Assume $s_{\min} < -l_0$ and $l_k(1) - l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \leq s_{\max}$ and $s_{\min} < l_k(0) - l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}$.*
(a) If $k = 0$,

$$l_0 + \ln \frac{1 - F_s(-l_0|\mathcal{R})}{1 - F_s(-l_0|\mathcal{L})} + \ln \frac{1 - F_s\left(-l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \Big| \mathcal{R}\right)}{1 - F_s\left(-l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \Big| \mathcal{L}\right)} + s_{\min} < 0, \quad (2)$$

$$l_0 + \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} + \ln \frac{F_s\left(-l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} \Big| \mathcal{R}\right)}{F_s\left(-l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} \Big| \mathcal{L}\right)} + s_{\max} \geq 0, \quad (3)$$

then a_3 is informative after any a^2.
(b) If $k > 0$,

$$l_0 + \ln \frac{1 - F_s(-l_0|\mathcal{R})}{1 - F_s(-l_0|\mathcal{L})} + \ln \frac{1 - F_s\left(l_k(1) - l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \Big| \mathcal{R}\right)}{1 - F_s\left(l_k(1) - l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \Big| \mathcal{L}\right)} + s_{\min} \geq l_k(1), \quad (4)$$

$$l_0 + \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} + \ln \frac{F_s\left(l_k(0) - l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} \Big| \mathcal{R}\right)}{F_s\left(l_k(0) - l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} \Big| \mathcal{L}\right)} + s_{\max} < l_k(0), \quad (5)$$

then agent 3 herds after $a_1 = a_2$.

Proof. Under $s_{\min} < l_k(1) - l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \leq s_{\max}$ and $s_{\min} < l_k(0) - l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} \leq s_{\max}$, agent 2 does not herd or anti-herd. Condition (2) implies $s_{\min} < l_k(1) - l_0 - \ln \frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}$, and (3) implies $l_k(0) - l_0 - \ln \frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} \leq s_{\max}$.

(a) Agent 3 chooses $a_3(R,R,s_{\min}) = L$ if (2) holds, and $a_3(R,R,s_{\max}) = R$ if $l_0 + \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} + \ln\frac{1-F_s\left(-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{1-F_s\left(-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)} + s_{\max} \geq l_k(1)$, which is implied by $l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} \leq s_{\max}$ (agent 2 not anti-herding after $a_1 = R$). Symmetric reasoning establishes $a_3(L,L,s_{\max}) = R$ if (3) holds, and $a_3(L,L,s_{\min}) = L$ if agent 2 does not anti-herd after L.

To show that $a_2 \neq a_1$ does not lead to herding: $a_3(R,L,s_{\min}) = L$ iff $l_0 + \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} + \ln\frac{F_s\left(-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{F_s\left(-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)} + s_{\min} < 0$, which is implied by $k = 0$ and $s_{\min} < l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}$ (agent 2 not herding after $a_1 = R$). Similarly, $a_3(R,L,s_{\max}) = R$ iff $l_0 + \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} + \ln\frac{F_s\left(-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{F_s\left(-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)} + s_{\max} \geq 0$, implied by $k = 0$ and (3).

(b) If (4) holds, then $a_3(R,R,s_{\min}) = R$, so $a_3(R,R,s) = R$ for any s. Under (5), $a_3(L,L,s_{\max}) = L$, so $a_3(L,L,s) = L$ for any s. □

Given Proposition 7 (a), sufficient for Proposition 7 (b) to be satisfied is that $s_{\max} - s_{\min}$ is large enough and the left sides of (4) and (5) change in $l_k(1)$ faster than unity, i.e. the negative hazard rate $\frac{-dF_s\left(l_k(1)-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{1-F_s\left(l_k(1)-l_0-\ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}$ of the signal distribution $F_s(\cdot|\mathcal{R})$ exceeds the hazard rate of $F_s(\cdot|\mathcal{L})$ by at least 1, and similarly $\frac{dF_s\left(-l_k(1)-l_0-\ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{F_s\left(-l_k(1)-l_0-\ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)} - \frac{dF_s\left(-l_k(1)-l_0-\ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)}{F_s\left(-l_k(1)-l_0-\ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)} > 1$. For example, $F(\cdot|\theta)$ are truncated normal distributions with the same support and variance, and the mean of $F(\cdot|\mathcal{R})$ exceeds that of $F(\cdot|\mathcal{L})$ by a large enough amount.

Intuitively, if agent 3's desire to differ is outweighed by the effect of 2's desire to differ on 3's belief updating given that 2 matches 1, then agent 3 herds more. After agent 2 mismatches 1, both the updating and the desire to differ reduce 3's herding motive, but if 3 did not herd under $k = 0$, then he cannot herd any less under $k > 0$.

****

Try: for any signal cdf F_s satisfying the MLR, bounded llr etc, there exist $k_1 < k_2$ s.t. for any $k \in [k_1, k_2]$, $\Pr(a_3 = \theta)$ increases in k. Forces: 1. If k is large, then $a_2 = a_1$ is probability zero, or so informative that 3 herds, but in either case, 3 does not herd more in response to higher k. 2. If k is large, then $a_2 \neq a_1$ almost uninformative, so $l_3(a_1 \neq a_2)$ has the same sign as $l_2(a_1)$. Then higher k increases $|l_3(a_1 \neq a_2)|$, so belief conditional on equal and unequal past actions becomes more extreme. More extreme bel increases 3's probability of matching the state.

If k is large enough s.t. $l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} > s_{\max}$ and $l_k(0) - l_0 - \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} < s_{\min}$, then for both θ,

$$\Pr(a_1 = a_2 = R|\theta) = [1 - F_s(-l_0|\theta)]\left[1 - F_s\left(l_k(1) - l_0 - \ln\frac{1 - F_s(-l_0|\mathcal{R})}{1 - F_s(-l_0|\mathcal{L})}\middle|\theta\right)\right] = 0,$$
$$\Pr(a_1 = a_2 = L|\theta) = F_s(-l_0|\theta)F_s\left(l_k(0) - l_0 - \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\theta\right) = 0.$$

The zero probability of $a_2 = a_1$ means that agent 2 anti-herds, which implies that a_2 is uninformative. Then agent 3's log likelihood ratio before seeing s_3 is $l_3(R, a_2) = l_0 + \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}$ after $a_1 = R$ and $l_3(L, a_2) = l_0 + \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}$ after $a_1 = L$ in the certain event $a_2 \neq a_1$. Since the two preceding agents take different actions, $f = \frac{1}{2}$ for agent 3. Therefore 3's best response is constant in k, as is 3's probability of matching the state $\Pr(a_3 = \theta) = p_0 \Pr(a_3 = R|\mathcal{R}) + (1 - p_0)[1 - \Pr(a_3 = R|\mathcal{L})]$, where

$$\Pr(a_3 = R|\theta) = [1 - F_s(-l_0|\theta)]\left[1 - F_s\left(-l_0 - \ln\frac{1 - F_s(-l_0|\mathcal{R})}{1 - F_s(-l_0|\mathcal{L})}\middle|\theta\right)\right] \tag{6}$$
$$+ F_s(-l_0|\theta)\left[1 - F_s\left(-l_0 - \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\theta\right)\right].$$

The probabilities in (6) are the same as for agent 2 under $k = 0$. Agent 2 is less likely than 3 to match the state under $k = 0$, because 3 obtains extra information by observing a_2 (the signals s_2, s_3 are equally informative and both 2 and 3 see the same a_1). Thus $\Pr(a_3 = \theta)$ is larger under $k = 0$ than when k is large enough to make 2 anti-herd, which is intuitive.

Alternatively, if (4) and (5) hold for $k \in [k_1, k_2]$, then agent 3 herds after $a_1 = a_2$, so the probability $\Pr(a_3 = \theta|a_2 = a_1)$ of 3 matching the state after $a_1 = a_2$ stays constant in k. Necessary for (4) and (5) is k below a cutoff, to avoid anti-herding.

Opposite of what I want: If $l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} = s_{\max}$, then $1 - F_s\left(l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\theta\right) = dF_s\left(l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\theta\right)$, so (4) becomes $l_0 + \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})} + s_{\max} + s_{\min} \geq s_{\max} + l_0 + \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}$. This never holds, so 3 never herds after $a_1 = a_2 = R$. A similar reasoning establishes that a_3 is informative after $a_1 = a_2 = L$ if $l_k(0) - l_0 - \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} = s_{\min}$.

Conditions for 3 to herd from Prop 7:

$$l_0 + \ln\frac{1 - F_s(-l_0|\mathcal{R})}{1 - F_s(-l_0|\mathcal{L})} + \ln\frac{1 - F_s\left(l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{1 - F_s\left(l_k(1) - l_0 - \ln\frac{1-F_s(-l_0|\mathcal{R})}{1-F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)} + s_{\min} \geq l_k(1),$$

$$l_0 + \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})} + \ln\frac{F_s\left(l_k(0) - l_0 - \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\mathcal{R}\right)}{F_s\left(l_k(0) - l_0 - \ln\frac{F_s(-l_0|\mathcal{R})}{F_s(-l_0|\mathcal{L})}\middle|\mathcal{L}\right)} + s_{\max} < l_k(0),$$

6 Normally distributed state and signals

In this section, the state is $\theta \in \mathbb{R}$, $\theta \sim N(0,1)$ instead of binary. Each agent $i \in \mathbb{N}$ chooses an action $a_i \in \mathbb{R}$ after observing a private signal $s_i = \theta + \epsilon_s \in \mathbb{R}$, $s_i|\theta \sim N\left(\theta, \frac{1}{\tau_s}\right)$. The log likelihood ratio of signals is unbounded, so actions are always informative, ruling out herding and anti-herding. Bayesian updating in response to only the private signal yields $\theta|s_i \sim N\left(\frac{\tau_s s_i}{\tau_s+1}, \frac{1}{\tau_s+1}\right)$. Agents do not perfectly observe the actions of previous deciders, unlike in the baseline model. Instead, a public signal $y_i = a_i + \epsilon_y$ is revealed, with $y_i|a_i \sim N\left(a_i, \frac{1}{\tau_y}\right)$.

Payoffs are $u_i(a_i, y^{i-1}, \theta) = -(a_i - \theta)^2 + k\frac{1}{i-1}\sum_{j<i}(a_i - y_j)^2$, reflecting both a benefit from matching the state and a desire to differ from past realisations of public signals. Interpreting y_j as the realised action when j intended to take action a_j, it is natural that agent $i > j$ cares about y_j rather than a_j. In financial markets, the price impact y_j matters for agent $i > j$, not the amount a_j that agent j traded.

Assume $k \in (0,1)$, i.e. matching the state is relatively more important than avoiding past public signals. This prevents the best response from diverging to $\pm\infty$.

Focus on equilibria where each agent's action is linear in his signal.[2] By implication, actions are also linear in the public signals, the actions and signals of previous agents and in the state: $a_i^* = \beta_{0i} + \beta_{si}s_i + \sum_{j<i}\beta_{ji}y_j$. From the viewpoint of agent $j \neq i$, $y_i = \beta_{0i} + \sum_{j<i}\beta_{ji}y_j + \beta_{si}(\theta + \epsilon_s) + \epsilon_y$, so $y_i|\theta \sim N\left(\beta_{0i} + \sum_{j<i}\beta_{ji}y_j + \beta_{si}\theta, \frac{\beta_{si}^2}{\tau_s} + \frac{1}{\tau_y}\right)$. To Bayesian update θ based on a public signal y_i, agents first subtract the commonly known 'bias' terms from y_i and rescale it to $z_i = \frac{1}{\beta_{si}}[y_i - \beta_{0i} - \sum_{j<i}\beta_{ji}y_j] \sim N\left(s_i, \frac{1}{\beta_{si}^2\tau_y}\right) = N\left(\theta, \frac{\beta_{si}^2\tau_y\tau_s}{\tau_s + \beta_{si}^2\tau_y}\right)$. Bayes' rule yields

$$\theta|z_i \sim N\left(\frac{\frac{\tau_s+\beta_{si}^2\tau_y}{\beta_{si}^2\tau_y\tau_s}z_i}{\frac{\tau_s+\beta_{si}^2\tau_y}{\beta_{si}^2\tau_y\tau_s}+1}, \frac{1}{\frac{\tau_s+\beta_{si}^2\tau_y}{\beta_{si}^2\tau_y\tau_s}+1}\right) = N\left(\frac{(\tau_s+\beta_{si}^2\tau_y)z_i}{\tau_s+\beta_{si}^2\tau_y+\beta_{si}^2\tau_y\tau_s}, \frac{\beta_{si}^2\tau_y\tau_s}{\tau_s+\beta_{si}^2\tau_y+\beta_{si}^2\tau_y\tau_s}\right),$$

$$\theta|z^i \sim N\left(\frac{\sum_{n=1}^{i}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}z_n}{1+\sum_{n=1}^{i}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}}, \frac{1}{1+\sum_{n=1}^{i}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}}\right).$$

After observing z^{i-1} and s_i, agent i's posterior belief is

$$\theta|(z^{i-1}, s_i) \sim N\left(\frac{\sum_{n=1}^{i-1}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}z_n + \tau_s s_i}{\tau_s+1+\sum_{n=1}^{i-1}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}}, \frac{1}{\tau_s+1+\sum_{n=1}^{i-1}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}}\right).$$

[2] The strategy space is not restricted. An agent who expects all others to play linear strategies has a unique best response, which is linear.

The best response of agent $i > 1$ to predecessors' strategies $a_j^* = \beta_{0j} + \beta_{sj}s_j + \sum_{n<j}\beta_{nj}y_n$ maximises $\mathbb{E}_\theta\left[u_i(a^i)|z^{i-1}, s_i\right] = -a_i^2 + 2a_i\mathbb{E}\left[\theta|z^{i-1}, s_i\right] - \mathbb{E}\left[\theta^2|z^{i-1}, s_i\right] + ka_i^2 - 2a_i\frac{k}{i-1}\sum_{j<i}y_j + \frac{k}{i-1}\sum_{j<i}y_j^2$. The FOC is $a_i = \frac{1}{1-k}\mathbb{E}\left[\theta|z^{i-1}, s_i\right] - \frac{k}{(1-k)(i-1)}\sum_{j<i}y_j$.

Agent 1 chooses $a_1 = \mathbb{E}\left[\theta|s_1\right] = \frac{\tau_s s_1}{\tau_s+1}$, so $\beta_{01} = 0$ and $\beta_{s1} = \frac{\tau_s}{\tau_s+1}$. Agent 2 chooses $a_2 = \frac{1}{1-k}\mathbb{E}\left[\theta|z_1, s_2\right] - \frac{k}{1-k}y_1 = \frac{1}{1-k}\frac{\tau_s+\beta_{s1}^2\tau_y}{\beta_{s1}^2\tau_y\tau_s(\tau_s+1)+\tau_s+\beta_{s1}^2\tau_y}z_1 + \frac{1}{1-k}\frac{\beta_{s1}^2\tau_y\tau_s^2}{\beta_{s1}^2\tau_y\tau_s(\tau_s+1)+\tau_s+\beta_{s1}^2\tau_y}s_2 - \frac{k}{1-k}y_1 = \frac{1}{1-k}\frac{\tau_s+\beta_{s1}^2\tau_y}{\beta_{s1}^2\tau_y\tau_s(\tau_s+1)+\tau_s+\beta_{s1}^2\tau_y}\frac{1}{\beta_{s1}}y_1 + \frac{1}{1-k}\frac{\beta_{s1}^2\tau_y\tau_s^2}{\beta_{s1}^2\tau_y\tau_s(\tau_s+1)+\tau_s+\beta_{s1}^2\tau_y}s_2 - \frac{k}{1-k}y_1 = \frac{1}{1-k}\frac{\tau_s+(\frac{\tau_s}{\tau_s+1})^2\tau_y}{(\frac{\tau_s}{\tau_s+1})^2\tau_y\tau_s(\tau_s+1)+\tau_s+(\frac{\tau_s}{\tau_s+1})^2\tau_y}\frac{\tau_s+1}{\tau_s}y_1 + \frac{1}{1-k}\frac{(\frac{\tau_s}{\tau_s+1})^2\tau_y\tau_s^2}{(\frac{\tau_s}{\tau_s+1})^2\tau_y\tau_s(\tau_s+1)+\tau_s+(\frac{\tau_s}{\tau_s+1})^2\tau_y}s_2 - \frac{k}{1-k}y_1 = \frac{1}{1-k}\frac{\tau_s(\tau_s+1)^2+\tau_s^2\tau_y}{\tau_s^2\tau_y\tau_s(\tau_s+1)+\tau_s(\tau_s+1)^2+\tau_s^2\tau_y}\frac{\tau_s+1}{\tau_s}y_1 - \frac{k}{1-k}y_1 + \frac{1}{1-k}\frac{\tau_s^2\tau_y\tau_s^2}{\tau_s^2\tau_y\tau_s(\tau_s+1)+\tau_s(\tau_s+1)^2}$ $\frac{1}{1-k}\left[\frac{(\tau_s+1)^3+\tau_s(\tau_s+1)\tau_y}{\tau_s^3(\tau_s+1)\tau_y+\tau_s(\tau_s+1)^2+\tau_s^2\tau_y} - k\right]y_1 + \frac{1}{1-k}\frac{\tau_s^3\tau_y}{\tau_s^2(\tau_s+1)\tau_y+(\tau_s+1)^2+\tau_s\tau_y}s_2$. Agent i's best response is

$$a_i = \frac{1}{1-k}\frac{\sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^3\tau_y\tau_s}[y_j - \beta_{0j} - \sum_{n<j}\beta_{nj}y_n] + \tau_s s_i}{\tau_s + 1 + \sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^2\tau_y\tau_s}} - \frac{k}{(1-k)(i-1)}\sum_{j<i}y_j. \tag{7}$$

By induction, $\beta_{0i} = 0$ for all i. In a_i in (7), the coefficient $\beta_{i-1,i} = \frac{1}{1-k}\frac{\frac{\tau_s+\beta_{s,i-1}^2\tau_y}{\beta_{s,i-1}^3\tau_y\tau_s}}{\tau_s+1+\sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^2\tau_y\tau_s}} - \frac{k}{(1-k)(i-1)}$, on y_{i-1} is a function of $(\beta_{sj})_{j=1}^{i-1}$ only, not of past β_{nj}, $n < j$, but the coefficients β_{ji} for $j \leq i-2$ are functions of β_{nj}, $n < j$. From $\beta_{s1} = \frac{\tau_s}{\tau_s+1}$ and the inductive formula $\beta_{si} = \frac{\tau_s}{(1-k)\left(\tau_s+1+\sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^2\tau_y\tau_s}\right)}$, we get $\beta_{s2} = \frac{\tau_s}{(1-k)\left(\tau_s+1+\frac{\tau_s+(\frac{\tau_s}{\tau_s+1})^2\tau_y}{(\frac{\tau_s}{\tau_s+1})^2\tau_y\tau_s}\right)} = \frac{\tau_s}{(1-k)\left[\tau_s+1+\frac{(\tau_s+1)^2+\tau_s\tau_y}{\tau_s^2\tau_y}\right]} = \frac{\tau_s^3\tau_y}{(1-k)[\tau_s^3\tau_y+\tau_s^2\tau_y+\tau_s\tau_y+(\tau_s+1)^2]}$ and $\beta_{s3} = \frac{\tau_s}{(1-k)\left(\tau_s+1+\frac{(\tau_s+1)^2+\tau_s\tau_y}{\tau_s^2\tau_y}+\frac{\tau_s(1-k)[\tau_s^3\tau_y+\tau_s^2\tau_y+\tau_s\tau_y+(\tau_s+1)^2]+\tau_s^6\tau_y^2\tau_y}{\tau_s^6\tau_y^2\tau_y\tau_s}\right)} = \frac{\tau_s^7\tau_y^3}{(1-k)\left(\tau_s^7\tau_y^3+\tau_s^6\tau_y^3+\tau_s^4\tau_y^2(\tau_s+1)^2+\tau_s^5\tau_y^3+(1-k)[\tau_s^3\tau_y+\tau_s^2\tau_y+\tau_s\tau_y+(\tau_s+1)^2]+\tau_s^5\tau_y^3\right)} = \frac{\tau_s^7\tau_y^3}{(1-k)\left(\tau_s^7\tau_y^3+\tau_s^6\tau_y^3+2\tau_s^5\tau_y^3+\tau_s^4\tau_y^2(\tau_s+1)^2+(1-k)[\tau_s^3\tau_y+\tau_s^2\right.}$

The inductive formula for β_{si} can be rewritten $(1-k)\left(1 + \frac{1}{\tau_s} + \frac{i-1}{\tau_s^2} + \frac{1}{\tau_y\tau_s}\sum_{j<i}\frac{1}{\beta_{sj}^2}\right) = \frac{1}{\beta_{si}}$. Even if just the immediate predecessor matters, the β_{si} are complicated.

$$\begin{aligned}\mathbb{E}_{\epsilon_s,\epsilon_y}(a_i-\theta)^2 &= \mathbb{E}_{\epsilon_s,\epsilon_y}\left(\frac{\sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^3\tau_y\tau_s}[\beta_{sj}\theta + \beta_{sj}\epsilon_s + \epsilon_y] + \tau_s\theta + \tau_s\epsilon_s}{(1-k)\left(\tau_s + 1 + \sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^2\tau_y\tau_s}\right)}\right. \\ &\quad \left. - \frac{k\sum_{j<i}(\beta_{sj}\theta + \beta_{sj}\epsilon_s + \epsilon_y + \sum_{n<j}\beta_{nj}y_n)}{(1-k)(i-1)} - \frac{\theta}{1-k} + \frac{k\theta(i-1)}{(1-k)(i-1)}\right)^2 \\ &= \mathbb{E}_{\epsilon_s,\epsilon_y}\left(\frac{\sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^3\tau_y\tau_s}[\beta_{sj}\theta + \beta_{sj}\epsilon_s + \epsilon_y] + \tau_s\theta + \tau_s\epsilon_s - \tau_s\theta - \theta - \theta\sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^2\tau_y\tau_s}}{(1-k)\left(\tau_s + 1 + \sum_{j<i}\frac{\tau_s+\beta_{sj}^2\tau_y}{\beta_{sj}^2\tau_y\tau_s}\right)}\right. \\ &\quad \left. - \frac{k\sum_{j<i}(\beta_{sj}\theta - \theta + \beta_{sj}\epsilon_s + \epsilon_y + \sum_{n<j}\beta_{nj}y_n)}{(1-k)(i-1)}\right)^2,\end{aligned}$$

where $\sum_{j<i} y_j = \sum_{j<i}\left[\beta_{sj}\theta + \beta_{sj}\epsilon_{sj} + \epsilon_{yj} + \sum_{n<j}\beta_{nj}\left(\beta_{sn}\theta + \beta_{sn}\epsilon_{sn} + \epsilon_{yn} + \sum_{m<n}\beta_{mn}y_m\right)\right] = \sum_{j<i}\beta_{sj}\theta + \sum_{j<i}\beta_{sj}\epsilon_{sj} + \sum_{j<i}\epsilon_{yj} + \sum_{j<i}\sum_{n<j}\beta_{nj}\beta_{sn}\theta + \sum_{j<i}\sum_{n<j}\beta_{nj}\beta_{sn}\epsilon_{sn} + \sum_{j<i}\sum_{n<j}\beta_{nj}\epsilon_{yn} + \sum_{j<i}\sum_{n<j}\beta_{nj}\sum_{m<n}\beta_{mn}\left(\beta_{sm}\theta + \beta_{sm}\epsilon_{sm} + \epsilon_{ym} + \sum_{l<m}\beta_{lm}y_l\right)$
$= \theta\sum_{j<i}\left(\beta_{sj} + \sum_{n<j}\beta_{nj}\beta_{sn} + \sum_{n<j}\beta_{nj}\sum_{m<n}\beta_{mn}\beta_{sm} + \ldots\right) + \sum_{j<i}\beta_{sj}\epsilon_{sj} + \sum_{j<i}\sum_{n<j}\beta_{nj}\beta_{sn}\epsilon_{sn} + \sum_{j<i}\sum_{n<j}\beta_{nj}\sum_{m<n}\beta_{mn}\beta_{sm}\epsilon_{sm} + \ldots + \sum_{j<i}\epsilon_{yj} + \sum_{j<i}\sum_{n<j}\beta_{nj}\epsilon_{yn} + \sum_{j<i}\sum_{n<j}\beta_{nj}\sum_{m<n}\beta_{mn}\epsilon_{ym} + \ldots + \sum_{j<i}\sum_{n<j}\beta_{nj}\sum_{m<n}\beta_{mn}\sum_{l<m}\beta_{lm}\left(\beta_{sl}\theta + \beta_{sl}\epsilon_{sl} + \epsilon_{yl} + \sum_{k<l}\beta_{kl}y_k\right)$. Find the β_{si}, β_{ji} coefficients to compare $k = 0$ and $k > 0$.

Is state-matching better with a desire to differ? In terms of expected quadratic loss, as in the utility function. Probably constant, because pooling more signals can only occur via a smaller slope of a_j in θ. With public y_j, seems that only the level of a_i changes in y_j, not the slope in θ.

Does the correlation between a_j, θ affect updating? Use matrixes: jointly normal independent primitives $\theta, \epsilon_{sj}, \epsilon_{yj}$, jointly normal observations z_j, s_i.

Robust comp stats if equations for β complicated.

Insider trading papers.

Gaussian, try to mismatch past actions, not public signals

In this section, the state is $\theta \in \mathbb{R}$, $\theta \sim N(0,1)$ instead of binary. Each agent $i \in \mathbb{N}$ chooses an action $a_i \in \mathbb{R}$ after observing a private signal $s_i = \theta + \epsilon_s \in \mathbb{R}$, $s_i|\theta \sim N\left(\theta, \frac{1}{\tau_s}\right)$. Bayesian updating in response to only the private signal yields $\theta|s_i \sim N\left(\frac{\tau_s s_i}{\tau_s+1}, \frac{1}{\tau_s+1}\right)$. Agents do not perfectly observe the actions of previous deciders, unlike in the baseline model. Instead, a public signal $y_i = a_i + \epsilon_y$ is revealed, with $y_i|a_i \sim N\left(a_i, \frac{1}{\tau_y}\right)$.

The jointly normal observations (z^{i-1}, s_i) are used to update $\theta, (\epsilon_{sj}, \epsilon_{yj})_{j=1}^{i-1}$.

Payoffs are $u_i(a^i, \theta) = -(a_i - \theta)^2 + k\frac{1}{i-1}\sum_{j<i}(a_i - a_j)^2$, reflecting both a benefit from matching the state and a desire to differ from previous actions. Assume $k \in (0,1)$, so matching the state is relatively more important, which prevents the best response from diverging to $\pm\infty$.

Focus on equilibria where each agent's action is linear in his signal.[3] By implication, actions are also linear in the public signals, the actions and signals of previous agents and in the state: $a_i^* = \beta_{0i} + \beta_{si}s_i + \sum_{j<i}\beta_{ji}y_j$. From the viewpoint of agent $j \neq i$, $y_i = \beta_{0i} + \sum_{j<i}\beta_{ji}y_j + \beta_{si}(\theta + \epsilon_s) + \epsilon_y$, so $y_i|\theta \sim N\left(\beta_{0i} + \sum_{j<i}\beta_{ji}y_j + \beta_{si}\theta,\ \frac{\beta_{si}^2}{\tau_s} + \frac{1}{\tau_y}\right)$. To Bayesian update based on a public signal y_i, agents first subtract the commonly known 'bias' terms from y_i and rescale it to $z_i = \frac{1}{\beta_{si}}[y_i - \beta_{0i} - \sum_{j<i}\beta_{ji}y_j] \sim N\left(s_i, \frac{1}{\beta_{si}^2\tau_y}\right) = N\left(\theta, \frac{\beta_{si}^2\tau_y\tau_s}{\tau_s + \beta_{si}^2\tau_y}\right)$.

[3] The strategy space is not restricted. An agent who expects all others to play linear strategies has a unique best response, which is linear.

Define $(z_1, \dots, z_i)$ by z^i. Bayes' rule yields

$$\theta|z_i \sim N\left(\frac{\frac{\tau_s+\beta_{si}^2\tau_y}{\beta_{si}^2\tau_y\tau_s}z_i}{\frac{\tau_s+\beta_{si}^2\tau_y}{\beta_{si}^2\tau_y\tau_s}+1},\ \frac{1}{\frac{\tau_s+\beta_{si}^2\tau_y}{\beta_{si}^2\tau_y\tau_s}+1}\right) = N\left(\frac{(\tau_s+\beta_{si}^2\tau_y)z_i}{\tau_s+\beta_{si}^2\tau_y+\beta_{si}^2\tau_y\tau_s},\ \frac{\beta_{si}^2\tau_y\tau_s}{\tau_s+\beta_{si}^2\tau_y+\beta_{si}^2\tau_y\tau_s}\right),$$

$$\theta|z^i \sim N\left(\frac{\sum_{n=1}^{i}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}z_n}{1+\sum_{n=1}^{i}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}},\ \frac{1}{1+\sum_{n=1}^{i}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}}\right).$$

After observing z^{i-1} and s_i, agent i's posterior belief is

$$\theta|(z^{i-1}, s_i) \sim N\left(\frac{\sum_{n=1}^{i-1}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}z_n+\tau_s s_i}{\tau_s+1+\sum_{n=1}^{i-1}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}},\ \frac{1}{\tau_s+1+\sum_{n=1}^{i-1}\frac{\tau_s+\beta_{sn}^2\tau_y}{\beta_{sn}^2\tau_y\tau_s}}\right).$$

The best response of agent $i > 1$ to previous movers' strategies $a_j^* = \beta_{0j} + \beta_{sj}s_j + \sum_{n<j}\beta_{nj}y_n$ maximises $\mathbb{E}_\theta\left[u_i(a^i)|z^{i-1}, s_i\right] = -a_i^2 + 2a_i\mathbb{E}\left[\theta|z^{i-1}, s_i\right] - \mathbb{E}\left[\theta^2|z^{i-1}, s_i\right] + ka_i^2 - 2a_i\frac{k}{i-1}\sum_{j<i}\mathbb{E}\left[a_j|z^{i-1}, s_i\right] + \frac{k}{i-1}\sum_{j<i}\mathbb{E}\left[a_j^2|z^{i-1}, s_i\right]$. Signals after a_j are indirectly informative about a_j via θ, so cannot just condition on z^j in $\mathbb{E}\left[a_j|z^{i-1}, s_i\right]$. The FOC is

$$a_i = \frac{1}{1-k}\mathbb{E}\left[\theta|z^{i-1}, s_i\right] - \frac{k}{(1-k)(i-1)}\sum_{j<i}\mathbb{E}\left[a_j|z^{i-1}, s_i\right]. \tag{8}$$

Agent 1 chooses $a_1 = \mathbb{E}\left[\theta|s_1\right] = \frac{\tau_s s_1}{\tau_s+1}$, so $\beta_{01} = 0$ and $\beta_{s1} = \frac{\tau_s}{\tau_s+1}$.

From agent 2's viewpoint, $a_1 = \frac{\tau_s}{\tau_s+1}(\theta+\epsilon_{s1})$, so the prior on a_1 is $N\left(0, \frac{\tau_s^2}{(\tau_s+1)^2}\left(1+\frac{1}{\tau_s}\right)\right) = N\left(0, \frac{\tau_s}{\tau_s+1}\right)$. The signal y_1 is distributed $N(a_1, \frac{1}{\tau_y})$ and the signal $s_2 = \theta+\epsilon_{s2} = \frac{\tau_s+1}{\tau_s}a_1 - \epsilon_{s1} + \epsilon_{s2}$ can be rescaled to $s_{a2} = a_1 - \frac{\tau_s}{\tau_s+1}(\epsilon_{s1}-\epsilon_{s2})$, which is distributed $N\left(a_1, \frac{\tau_s^2}{(\tau_s+1)^2}\frac{2}{\tau_s}\right)$. Potential problem: θ, a_1 are correlated. Does the additive utility make this correlation irrelevant? Agent 2 chooses $a_2 = \frac{1}{1-k}\mathbb{E}\left[\theta|z_1, s_2\right] - \frac{k}{1-k}\mathbb{E}\left[a_1|z_1, s_2\right] = \frac{\frac{\tau_s+\beta_{s1}^2\tau_y}{\beta_{s1}^2\tau_y\tau_s}z_1+\tau_s s_2}{\tau_s+1+\frac{\tau_s+\beta_{s1}^2\tau_y}{\beta_{s1}^2\tau_y\tau_s}} - \frac{k}{1-k}\cdots$

incomplete, questionable Prior joint distribution of agent 2, using $y_1 = \beta_{s1}(\theta+\epsilon_{s1})+\epsilon_{y1}$ and $s_2 = \theta+\epsilon_{s2}$:

$$\begin{bmatrix}\theta\\ \epsilon_{s1}\\ \epsilon_{y1}\\ \epsilon_{s2}\\ y_1\\ s_2\end{bmatrix} \sim N\left(0_{1\times 6},\ \begin{bmatrix}1 & 0 & \dots & 0 & & 1\\ 0 & \frac{1}{\tau_s} & 0 & 0 & & \\ \vdots & 0 & \frac{1}{\tau_y} & 0 & \frac{1}{\tau_y} & \\ 0 & & 0 & \frac{1}{\tau_s} & 0 & \frac{1}{\tau_s}\\ & & \frac{1}{\tau_y} & 0 & \beta_{s1}^2\left(1+\frac{1}{\tau_s}\right)+\frac{1}{\tau_y} & 0\\ 1 & & \dots & \frac{1}{\tau_s} & 0 & 1+\frac{1}{\tau_s}\end{bmatrix}\right)$$

The prior joint distribution on the observables $\vec{\omega}_2 := (y_1, s_2)^T$ and the variables of interest $\vec{\iota}_2 := (\theta, a_1)^T$ for agent 2 is derived from primitive uncertainty $\vec{\rho}_2 := (\theta, \epsilon_{s1}, \epsilon_{y1}, \epsilon_{s2})^T$ using

$y_1 = a_1 + \epsilon_{y1} = \beta_{s1}(\theta + \epsilon_{s1}) + \epsilon_{y1}$ and $s_2 = \theta + \epsilon_{s2}$. In matrix form, ordering the variables by the timing of their realisations,

$$\begin{bmatrix} \theta \\ a_1 \\ y_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_{s1} & \beta_{s1} & 0 & 0 \\ \beta_{s1} & \beta_{s1} & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \epsilon_{s1} \\ \epsilon_{y1} \\ \epsilon_{s2} \end{bmatrix} =: B_2\vec{\rho}_2,$$

where $\beta_{s1} = \frac{\tau_s}{\tau_s+1}$. For agents $i > 2$, the matrix B_i converting primitive uncertainty to the variables of interest and the observations is $2i \times 2i$, because the length of $\vec{\rho}_i$ is $1+2(i-1)+1 = 2i$ and the length of $\vec{\gamma}_i := (\theta, a^{i-1}, y^{i-1}, s_i)^T$ is also $2i$. The prior means of $\vec{\iota}_i, \vec{\omega}_i$ from agent i's viewpoint are denoted $\vec{\mu}_{i\iota}, \vec{\mu}_{i\omega}$; both equal zero. The prior covariance matrix of $\vec{\rho}_i$ is $C_{\rho i} := diag\left(1, \left(\frac{1}{\tau_s}, \frac{1}{\tau_y}\right)_{i=1}^{i-1}, \frac{1}{\tau_s}\right)$, because the components of $\vec{\rho}_i$ are independent.

The covariance matrix $Cov(\vec{\gamma}_2)$ of $\vec{\gamma}_2$ is $C_2 := B_2 C_{\rho 2} B_2^T$, or in longer notation

$$C_2 := Cov\begin{bmatrix} \theta \\ a_1 \\ y_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_{s1} & \beta_{s1} & 0 & 0 \\ \beta_{s1} & \beta_{s1} & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\tau_s} & 0 & 0 \\ 0 & 0 & \frac{1}{\tau_y} & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_s} \end{bmatrix} \begin{bmatrix} 1 & \beta_{s1} & \beta_{s1} & 1 \\ 0 & \beta_{s1} & \beta_{s1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_{s1} & \frac{1}{\tau_s}\beta_{s1} & 0 & 0 \\ \beta_{s1} & \frac{1}{\tau_s}\beta_{s1} & \frac{1}{\tau_y} & 0 \\ 1 & 0 & 0 & \frac{1}{\tau_s} \end{bmatrix} \begin{bmatrix} 1 & \beta_{s1} & \beta_{s1} & 1 \\ 0 & \beta_{s1} & \beta_{s1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} & 1 \\ \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} \\ \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} + \frac{1}{\tau_y} & \frac{\tau_s}{\tau_s+1} \\ 1 & \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} & \frac{\tau_s+1}{\tau_s} \end{bmatrix}.$$

Partition C_2 into $\begin{bmatrix} C_{2\iota\iota} & C_{2\iota\omega} \\ C_{2\omega\iota} & C_{2\omega\omega} \end{bmatrix}$, where all submatrices are square. The conditional distribution of $\vec{\iota}_2$ given $\vec{\omega}_2$ has mean $\vec{\mu}_{2\iota}|\vec{\omega}_2 := \vec{\mu}_{2\iota} + C_{2\iota\omega}C_{2\omega\omega}^{-1}(\vec{\omega}_2 - \vec{\mu}_{2\omega}) = C_{2\iota\omega}C_{2\omega\omega}^{-1}\vec{\omega}_2$ and posterior covariance $C_{2\iota\iota}^p := C_{2\iota\iota} - C_{2\iota\omega}C_{2\omega\omega}^{-1}C_{2\omega\iota}$ independent of the realised $\vec{\omega}_2$.

$$C_{2\omega\omega}^{-1} = \begin{bmatrix} \frac{\tau_s}{\tau_s+1} + \frac{1}{\tau_y} & \frac{\tau_s}{\tau_s+1} \\ \frac{\tau_s}{\tau_s+1} & \frac{\tau_s+1}{\tau_s} \end{bmatrix}^{-1} = \frac{1}{\left(\frac{\tau_s}{\tau_s+1} + \frac{1}{\tau_y}\right)\frac{\tau_s+1}{\tau_s} - \left(\frac{\tau_s}{\tau_s+1}\right)^2} \begin{bmatrix} \frac{\tau_s+1}{\tau_s} & -\frac{\tau_s}{\tau_s+1} \\ -\frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} + \frac{1}{\tau_y} \end{bmatrix} \text{ and}$$

$$C_{2\iota\omega}C_{2\omega\omega}^{-1} = \begin{bmatrix} \frac{\tau_s}{\tau_s+1} & 1 \\ \frac{\tau_s}{\tau_s+1} & \frac{\tau_s}{\tau_s+1} \end{bmatrix} C_{2\omega\omega}^{-1} = \frac{1}{1 + \frac{\tau_s+1}{\tau_y\tau_s} - \left(\frac{\tau_s}{\tau_s+1}\right)^2} \begin{bmatrix} 1 - \frac{\tau_s}{\tau_s+1} & -\left(\frac{\tau_s}{\tau_s+1}\right)^2 + \frac{\tau_s}{\tau_s+1} + \frac{1}{\tau_y} \\ 1 - \left(\frac{\tau_s}{\tau_s+1}\right)^2 & \frac{\tau_s}{\tau_y(\tau_s+1)} \end{bmatrix}.$$

$$\mathbb{E}\left[\begin{array}{c}\theta\\ a_1\end{array}\middle|\vec{\omega}_2\right] = \frac{1}{1+\frac{\tau_s+1}{\tau_y\tau_s}-\left(\frac{\tau_s}{\tau_s+1}\right)^2}\left[\begin{array}{c}\left(1-\frac{\tau_s}{\tau_s+1}\right)y_1+\left(\frac{\tau_s}{(\tau_s+1)^2}+\frac{1}{\tau_y}\right)s_2\\ \left(1-\left(\frac{\tau_s}{\tau_s+1}\right)^2\right)y_1+\frac{\tau_s}{\tau_y(\tau_s+1)}s_2\end{array}\right].$$ Using Mathematica,

$$a_2 = \frac{\left(1-k-\frac{\tau_s}{\tau_s+1}+k\left(\frac{\tau_s}{\tau_s+1}\right)^2\right)y_1+\left(\frac{\tau_s}{(\tau_s+1)^2}+\frac{1}{\tau_y}-k\frac{\tau_s}{\tau_y(\tau_s+1)}\right)s_2}{(1-k)\left[1+\frac{\tau_s+1}{\tau_y\tau_s}-\left(\frac{\tau_s}{\tau_s+1}\right)^2\right]}$$

$$= \frac{\tau_s\left[\tau_s\tau_y+(\tau_s+1)(1+\tau_s-k\tau_s)\right](\theta+\epsilon_{s2})+\tau_s\tau_y(1-k+\tau_s-2k\tau_s)\left(\frac{\tau_s}{\tau_s+1}\theta+\frac{\tau_s}{\tau_s+1}\epsilon_{s1}+\epsilon_{y1}\right)}{(1-k)\left[(1+\tau_s)^3+\tau_s\tau_y+2\tau_s^2\tau_y\right]}.$$

$$\frac{da_2}{d\theta} = \frac{\tau_s\left[(\tau_s+1)(1+\tau_s-k\tau_s)+\tau_s\tau_y+\frac{\tau_s\tau_y(1-k+\tau_s-2k\tau_s)}{\tau_s+1}\right]}{(1-k)\left[(1+\tau_s)^3+\tau_s\tau_y+2\tau_s^2\tau_y\right]} > 0.$$

$$\frac{d^2a_2}{d\theta dk} = \frac{\tau_s\left[1+2\tau_s+\tau_s^2+\tau_s\tau_y\right]}{(1-k)^2\left[(\tau_s+1)^4+\tau_s\tau_y(\tau_s+1)(1+2\tau_s)\right]} > 0.$$

So a_2 becomes more informative about θ when k increases.

The mean squared error $\mathbb{E}_{\theta,\epsilon_{s1},\epsilon_{y1},\epsilon_{s2}}(a_2-\theta)^2$ of a_2 increases in k by Mathematica.

For agent 3, the observables as functions of primitive uncertainty are $y_1 = \beta_{s1}(\theta+\epsilon_{s1})+\epsilon_{y1}$, $y_2 = \beta_{12}y_1+\beta_{s2}(\theta+\epsilon_{s2})+\epsilon_{y2} = (\beta_{12}\beta_{s1}+\beta_{s2})\theta+\beta_{12}\beta_{s1}\epsilon_{s1}+\beta_{12}\epsilon_{y1}+\beta_{s2}\epsilon_{s2}+\epsilon_{y2}$ and $s_3 = \theta+\epsilon_{s3}$. The unobservable a_2 from agent 3's viewpoint is $a_2 = \beta_{12}\beta_{s1}\theta+\beta_{12}\beta_{s1}\epsilon_{s1}+\beta_{12}\epsilon_{y1}+\beta_{s2}(\theta+\epsilon_{s2})$. Therefore the equation $\vec{\gamma}_3 = B_3\vec{\rho}_3$ is

$$\begin{bmatrix}\theta\\ a_1\\ a_2\\ y_1\\ y_2\\ s_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ \beta_{s1} & \beta_{s1} & 0 & 0 & 0 & 0\\ \beta_{12}\beta_{s1}+\beta_{s2} & \beta_{12}\beta_{s1} & \beta_{12} & \beta_{s2} & 0 & 0\\ \beta_{s1} & \beta_{s1} & 1 & 0 & 0 & 0\\ \beta_{12}\beta_{s1}+\beta_{s2} & \beta_{12}\beta_{s1} & \beta_{12} & \beta_{s2} & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta\\ \epsilon_{s1}\\ \epsilon_{y1}\\ \epsilon_{s2}\\ \epsilon_{y2}\\ \epsilon_{s3}\end{bmatrix},$$

where $\beta_{s1} = \frac{\tau_s}{\tau_s+1}$, $\beta_{12} = \frac{da_2}{dy_1}$ and $\beta_{s2} = \frac{da_2}{ds_2}$ have been found previously.

The mean squared error $\mathbb{E}_{\vec{\rho}_3}(a_3-\theta)^2$ of a_3 increases in k by Mathematica.

Maybe $\mathbb{E}(a_i-\theta)^2$ can be incorporated as one more step in the matrixes, without simplifying the algebra?

For agent i, have to invert an $i\times i$ matrix.

7 Discussion

The result that herding may increase with the desire to differ from previous movers is robust to varying the informativeness of signals or the congestion cost within some bounds. The informativeness and cost may also differ to some extent across players, and players may be somewhat uncertain of others' preferences and signal precisions without changing the results. Unboundedly informative signals or a strong enough preference for non-conformity break herding, as established in the previous literature. If the congestion cost is small enough, then it does not affect players' actions, because it does not outweigh the weakest of the finitely many signals.

In some applications, the congestion cost depends only on the actions of some preceding agents, not all. For example, if a service provider is capacity constrained and can serve only m agents at a time or finishes the service in at most m periods, then an agent's payoff only depends on the choices of the m immediate predecessors. The desire to differ may increase conformity also in this case, as is clear from redefining f in Section 2 to be the fraction of agents among the preceding m who choose R.

Even if congestion depends only on the immediately preceding agent, a more informative $a_2 = a_1$ can motivate player 3 to herd. The less informative $a_2 \neq a_1$ cannot reduce player 3's herding compared to the $k = 0$ case if 3 does not herd after $a_2 \neq a_1$ under $k = 0$. Thus the overall probability of herding may increase, as in the baseline model. The proofs simplify, because each time the belief returns to the prior, the subgame is identical to the whole game. In particular, the condition $l_0 + l_{Qq} - l_q - l_k\left(\frac{i+1}{2i+1}\right) < 0$ in Proposition 2 for herding to start in period $2i+3$ conditional on not having started earlier may be omitted w.l.o.g., because it reduces to $l_0 + l_{Qq} - l_q - l_k\left(1\right) < 0$.

Qualitatively similar results also obtain if congestion depends on a discounted (or otherwise weighted) average of the actions of previous movers. Again, player 3 faces the same problem as in the baseline model, and the problems of subsequent odd players only differ in the effective congestion cost.

The results resemble the baseline model when agents do not observe their immediate predecessor, but see all other previous movers (as in one model of Eyster and Rabin (2014)). In this case, agent 2 faces the same decision as agent 1 in the baseline environment. Agent 3 solves the same problem as 2 originally. Agent 4 observes the (now independent) actions by 1 and 2, so has the same information as 3 in a situation without congestion costs. Thus 4 may herd due to the standard informational reason if the desire to differ is weak enough. If agent 5 observed only 1 and 3, then he would be exactly in the position of 3 in the baseline model. The action of 2 provides an extra signal to 5, which may increase or decrease herding by 5, depending on whether 2's action coincides with that of 1 or not. However, a desire to differ induces 5 to herd more response to equal actions of 1 and 3.

A Herding reduced by the desire to conform

This section shows that a preference to match the actions of preceding agents may in fact reduce herding. The idea is similar to why the desire to differ may increase herding—the actions of previous players become more informative after some histories, less after others. A strong signal overwhelms the effect of two previous less informative actions plus the desire to conform, but does not outweigh the more informative actions in the absence of a preference to follow previous movers.

Only the differences from the setup in Section 1 are mentioned. Payoffs are

$$u_i(a^i,\theta) = \mathbf{1}\left\{a_i = \theta\right\} + \frac{k}{i-1}\sum_{j=1}^{i-1}\mathbf{1}\left\{a_j = a_i\right\},$$

where $k \geq 0$ as before, but here the payoff from an action increases in the fraction f of previous agents taking that action.

There are six possible signal realisations $s_i \in \{\ell_\alpha, \ell_\beta, \ell_\gamma, r_\gamma, r_\beta, r_\alpha\}$, ordered from strongest in favour of $\theta = \mathcal{L}$ to the strongest favouring $\mathcal{R}$. The respective unconditional probabilities of a strong, medium and weak signal are $p_\alpha := \Pr(\ell_\alpha) + \Pr(r_\alpha)$, $p_\beta := \Pr(\ell_\beta) + \Pr(\ell_\beta)$ and $p_\gamma := \Pr(\ell_\gamma) + \Pr(r_\gamma)$. The conditional probabilities are $\Pr(\ell_\alpha|\mathcal{L}) = \Pr(r_\alpha|\mathcal{R}) =: q_\alpha$, $\Pr(\ell_\beta|\mathcal{L}) = \Pr(r_\beta|\mathcal{R}) =: q_\beta$ and $\Pr(\ell_\gamma|\mathcal{L}) = \Pr(r_\gamma|\mathcal{R}) =: q_\gamma$. Assume $\frac{1}{2} < \frac{q_\gamma}{p_\gamma} < \frac{q_\beta}{p_\beta} < \frac{q_\alpha}{p_\alpha} < 1$, which justifies the interpretations of the signals. Define

$l_x := \ln q_x - \ln(p_x - q_x)$ for $x \in \{\alpha, \beta, \gamma\}$,

$l_{\alpha\beta} := \ln(q_\alpha + q_\beta) - \ln(p_\alpha + p_\beta - q_\alpha - q_\beta) \in (l_\beta, l_\alpha)$,

$l_{\alpha\beta\gamma} := \ln(q_\alpha + q_\beta + q_\gamma) - \ln(1 - q_\alpha - q_\beta - q_\gamma) \in (l_\gamma, l_\alpha)$,

$l_{\neg\beta\alpha} := \ln(p_\gamma + q_\beta + q_\alpha) - \ln(1 - q_\beta - q_\alpha) \in (0, l_{\alpha\beta\gamma})$,

$l_{\neg\alpha} := \ln(p_\beta + p_\gamma + q_\alpha) - \ln(1 - q_\alpha) \in (0, l_{\neg\beta\alpha})$.

The next result is analogous to Proposition 2 and provides sufficient conditions for herding to decrease when conformism is introduced.

Proposition 9. *Assume $l_\gamma > l_0$. If $k = 0$, $l_0 + l_{\alpha\beta\gamma} - l_\beta < 0$, $l_0 - l_{\alpha\beta\gamma} + l_\gamma < 0$ and $l_0 - l_{\alpha\beta\gamma} - l_{\neg\beta\alpha} + l_\alpha < 0$, then a_3 is uninformative after $a_2 = a_1$, the probability of which is $(q_\alpha + q_\beta + q_\gamma)(p_\gamma + q_\beta + q_\alpha) + (1 - q_\alpha - q_\beta - q_\gamma)(1 - q_\alpha - q_\beta)$.*

If $k > 0$, $l_0 - l_{\alpha\beta\gamma} + l_\beta - l_k(1) < 0$ and $l_0 + l_{\alpha\beta\gamma} + l_{\neg\alpha} - l_\alpha + l_k(1) < 0$, then a_3 is informative after any history.

Proof. The assumption $l_0 < l_\gamma$ ensures that player 1 follows own signal. Then player 2's public log likelihood ratios are $l_2(L) = l_0 - l_{\alpha\beta\gamma}$ and $l_2(R) = l_0 + l_{\alpha\beta\gamma}$.

$k = 0$. Assume $l_0 + l_{\alpha\beta\gamma} - l_\beta < 0$ and $l_0 - l_{\alpha\beta\gamma} + l_\gamma < 0$, so $a_2(R, \ell_\beta) = a_2(L, r_\gamma) = L$ and by implication, $a_2(L, r_\beta) = a_2(R, \ell_\gamma) = R$. Player 3's log likelihood ratios before seeing s_3 are

$$l_3(L, L) = l_0 - l_{\alpha\beta\gamma} - l_{\neg\beta\alpha}, \qquad l_3(L, R) = l_0 - l_{\alpha\beta\gamma} + l_{\alpha\beta},$$
$$l_3(R, L) = l_0 + l_{\alpha\beta\gamma} - l_{\alpha\beta}, \qquad l_3(R, R) = l_0 + l_{\alpha\beta\gamma} + l_{\neg\beta\alpha}.$$

If $l_0 - l_{\alpha\beta\gamma} - l_{\neg\beta\alpha} + l_\alpha < 0$, then a_3 is uninformative after $a_2 = a_1$, i.e. a herd starts. After $a_2 \neq a_1$, player 3's action always responds to signals.

$k > 0$. If $l_0 - l_{\alpha\beta\gamma} + l_\beta - l_k(1) < 0$ and $l_0 + l_{\alpha\beta\gamma} - l_\alpha + l_k(1) < 0$ (which is implied by $l_0 + l_{\alpha\beta\gamma} + l_{\neg\alpha} - l_\alpha + l_k(1) < 0$), then $a_2(L, r_\beta) = a_2(R, \ell_\alpha) = L$ and by implication, $a_2(R, \ell_\beta) = a_2(L, r_\alpha) = R$. Player 3's log likelihood ratios before observing s_3 are then

$$l_3(L, L) = l_0 - l_{\alpha\beta\gamma} - l_{\neg\alpha}, \qquad l_3(L, R) = l_0 - l_{\alpha\beta\gamma} + l_\alpha,$$
$$l_3(R, L) = l_0 + l_{\alpha\beta\gamma} - l_\alpha, \qquad l_3(R, R) = l_0 + l_{\alpha\beta\gamma} + l_{\neg\alpha}.$$

If $l_0 + l_{\alpha\beta\gamma} + l_{\neg\alpha} - l_\alpha + l_k(1) < 0$, then a strong signal switches the sign of $l_3(R, R)$, so a_3 is informative after $a_2 = a_1 = R$ and by implication after any history. □

The next example exhibits parameter values satisfying the assumptions of Proposition 8.

Example 2. Let $l_0 = 0$, $p_\alpha = \frac{4}{5}$, $p_\beta = p_\gamma = \frac{1}{10}$, $\frac{q_\alpha}{p_\alpha} \approx 0.984$, $\frac{q_\beta}{p_\beta} \approx 0.93$, $\frac{q_\gamma}{p_\gamma} \approx 0.5$ and $k \approx 1.9 \cdot 10^{-6}$, or alternatively $p_0 = 0.51$, $p_\alpha = \frac{4}{5}$, $p_\beta = p_\gamma = \frac{1}{10}$, $\frac{q_\alpha}{p_\alpha} \approx 0.987$, $\frac{q_\beta}{p_\beta} \approx 0.935$, $\frac{q_\gamma}{p_\gamma} \approx 0.5002$ and $k \approx 0.04$.

The probability of $a_2 = a_1$ is $(q_\alpha + q_\beta + q_\gamma)(p_\gamma + q_\beta + q_\alpha) + (1 - q_\alpha - q_\beta - q_\gamma)(1 - q_\beta - q_\alpha) \approx 0.92$ under $k = 0$, but $(q_\alpha + q_\beta + q_\gamma)(p_\gamma + p_\beta + q_\alpha) + (1 - q_\alpha - q_\beta - q_\gamma)(1 - q_\alpha) \approx 0.94$ under $k > 0$. Thus in both examples, the probability that the action of player 3 is informative rises from about 0.08 to 1 when the desire to conform is introduced.

The desire to conform is related to Callander (2007), which assumes voters choosing one of two candidates desire to vote with the majority, as well as pick the better candidate. The current paper instead assumes that deciders wish to match only past agents' actions, not anticipated future choices. In Callander (2007), Examples 1, 2 and Theorems 4, 5 show that a desire to conform can increase herding, as is intuitive. With naive beliefs, conformism can also reduce herding (Corollary 1), but results in uninformative oscillation, unlike in the present work.

References

Ali, S. N. and N. Kartik (2012): "Herding with collective preferences," *Economic Theory*, 51, 601–626.

Banerjee, A. V. (1992): "A simple model of herd behavior," *The Quarterly Journal of Economics*, 107, 797–817.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992): "A theory of fads, fashion, custom, and cultural change as informational cascades," *Journal of Political Economy*, 100, 992–1026.

Callander, S. (2007): "Bandwagons and momentum in sequential voting," *The Review of Economic Studies*, 74, 653–684.

Callander, S. and J. Hörner (2009): "The wisdom of the minority," *Journal of Economic Theory*, 144, 1421–1439.

DeMarzo, P. M., D. Vayanos, and J. Zwiebel (2003): "Persuasion bias, social influence, and unidimensional opinions," *The Quarterly journal of economics*, 118, 909–968.

Eyster, E., A. Galeotti, N. Kartik, and M. Rabin (2014): "Congested observational learning," *Games and Economic Behavior*, 87, 519–538.

Eyster, E. and M. Rabin (2010): "Naïve Herding in Rich-Information Settings," *American Economic Journal: Microeconomics*, 2, 221–243.

——— (2014): "Extensive imitation is irrational and harmful," *The Quarterly Journal of Economics*, 129, 1861–1898.

Gaigl, A. (2009): "Observational Learning & Strategic Externalities," Working paper, University of Mannheim.

Smith, L. and P. Sørensen (2000): "Pathological outcomes of observational learning," *Econometrica*, 68, 371–398.