Growth-Maximizing Public Debt under Changing Demographics*

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Abstract: This paper develops an overlapping generations model to study the growth-maximizing level of public debt under conditions of demographic change. It is shown that the optimal debt level depends on a positive marginal productivity of public capital. It also depends on the demographic parameters, but not if the government is not allowed to borrow to cover revenue shortfalls for current age-related spending. Balanced budget rules, a key element in Europe’s fiscal compact, are therefore not an appropriate form of fiscal rule. The implication is that governments facing demographic change, or demands for more social spending, will have to adjust their fiscal plans to accommodate those changes.

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1 Introduction

Public debt and ageing populations are the two of the most pressing longer term problems facing all developed OECD economies. Recent debates have generated a lot of heat, but not much light between the conflicting policy recommendations of austerity vs. growth as strategies for reducing fiscal imbalances. This paper evaluates an alternative strategy: optimal debt targets, set to maximise growth and expressed as a function of population parameters and age-related spending. This approach allows us to assess whether the projected demographic changes call for even greater cuts in public debt than those already made necessary to resolve the sovereign debt crisis – especially in Europe.

Fiscal rules are not new to the European Union. The Maastricht Treaty imposed numerical limits on the fiscal debt and deficits of those joining the Euro area. Those limits became permanent with the Stability and Growth Pact in 1997. The most recent revision of the fiscal framework appears in the fiscal compact agreed in 2012. This is a balanced budget amendment requiring the structural fiscal balance of the general government to be held to a target value of zero, with a maximum structural deficit of 0.5 % of GDP.

In this paper we consider a different aspect, namely whether fiscal rules should be stated in terms of deficit or debt targets. We have argued elsewhere that debt targets are superior to deficit targets for both theoretical and practical reasons (Hughes Hallett and Jensen, 2012). Clearly, if a debt targeting rule is preferred, the question arises: what public debt level, or debt-to-GDP ratio, should be targeted? For example, is there such a thing as an optimal level of public debt? The answer may involve complicated trade-offs, such as how concerns about intergenerational equity should be balanced against economic performance and long term fiscal sustainability (Auerbach, 2009).

The problem of choosing a target for public debt also involves the question about whether to account for implicit (future) liabilities. Typically, the government liabilities entering the calculation of public debt only include explicit liabilities. However, if the implicit liabilities originating from ageing populations are ignored it means the debt criterion will ignore the future revenues required to avoid default despite the need to cover the benefits promised to existing workers/beneficiaries. This is the case for extending debt targeting rules to account for predictable demographic changes.

Several papers, including Kotlikoff (2006) and Davig et al. (2010), have emphasised the fiscal “overhang” posed by the uncovered expected financial liabilities associated with public pension schemes and likely health and social support costs in most OECD economies. A recent paper by
IMF (2009) has put this problem into perspective by showing that the financial stress caused by the great financial crash of 2007-10 was probably only about 10% of that likely to be caused by future age related spending in economies with a shrinking labour force. Against that, if fiscal sustainability is now the objective, it makes sense to search for fiscal rules capable of ensuring the sustainability of public finances given ageing populations, shrinking labour forces and greater implicit liabilities.

In this paper, we make three contributions. First, drawing on the idea of fiscal space, we offer a formal evaluation of the optimal debt level around which the economy needs to stabilise. ¹ Second, using an overlapping generations model ad modum Yakita (2008), we study how the optimal debt level will vary as a function of the population parameters. Third, we show how fiscal rules designed to stabilise the economy around that debt level need to vary with the population age, life expectancy, birth rate and rising welfare expenditures.

We find that balanced budget rules, key to Europe’s fiscal compact, are not appropriate. Instead, more sophisticated fiscal rules are needed which allow for the liabilities implied by population parameters and age-related spending. The implication is that a government facing demographic change or the need for higher social spending, will have to adjust (and most probably restrict) their fiscal plans to accommodate those changes. The fiscal space analysis provides a framework that allows the government to do so in a forward looking way and hence adjust to the new situation. A key theme of this paper is therefore to make debt control properly forward looking, by designing a rule where the instrument of fiscal policy reacts not only to changes in the existing level of debt but also to changes in future liabilities caused by changing demographics.

In principle, a debt targeting rule focused on the existing level of debt produces some forward looking behaviour because the debt stock is persistent: being off target today means being off target tomorrow too and hence more correction is needed today. However, such a rule will not allow for the additional burden caused by changes in demographics since these liabilities still lie in the future. And a fixed deficit limit rule, such as a balanced budget rule or the Euro Area’s fiscal compact, completely fails to account for predictable future liabilities in either sense.

To fix this omission, we add those features using an overlapping generations model to show the effect of population change on fiscal balances or debt. To our knowledge, there are only a few

¹The optimal debt level depends on the marginal productivity of public capital, which is defined as public spending on investment projects which (a) are productive, (b) have an identifiable rate of return and (c) have a longer investment horizon than consumption expenditures. It includes education therefore. Conceptually this is clear cut, but in practice there may be problems in measuring this separately from public consumption.
models that determine optimal levels of debt in the literature: Aschauer (2000) and Aiyagari and McGrattan (1998). Building on those models, we examine public debt under balanced growth, and where optimal government debt is defined to be that level which maximises growth.\footnote{There are empirical representations however: Reinhart and Rogoff (2010) and Checherita and Rother (2012). Also others where debt is defined to be unproductive: Greiner (2010). Aschauer (2000) considers a social optimum by maximizing the future stream of consumption spending, but finds that it makes little numerical difference. To keep things simple, we do not pursue that option. Auerbach (2009) contains a fuller discussion of the long-term objectives for public debt.}

Given this, we then ask: whether, and in what circumstances, deteriorating demographics will affect the net fiscal position; and, second, whether it is acceptable to allow larger debt burdens, or whether greater tax or spending austerity is always needed in such cases.

2 The model

The model that follows is based on Yakita’s (2008) overlapping generations model with population ageing, and public capital accumulation as the driving force behind economic growth. Specifically, we assume an overlapping generations economy populated by homogenous individuals, symmetric firms and a government. Individuals are assumed to have a lifespan divided into a working and a retirement period, respectively. In each period \( t \) a generation working in that period forms the young cohort. Their working lifetime is postulated to be of fixed length, whereas the retirement period is of uncertain length. At the end of each period a certain number of individuals die. We assume for simplicity that the individual is alive at the beginning of the next period with probability \( (1-\lambda) \in (0,1) \). This probability is the same for every individual. Individuals who survive into the second period of life are retired, and the generation of retirees forms the old cohort. Denoting the population of young and working agents as \( N_t \), the total population in period \( t \) is equal to \( N_t+(1-\lambda)N_{t-1} \).

2.1 Firms

We assume a large number of symmetric firms which produce a homogenous product by combining the services of private capital and labor. The production technology for each firm \( j \) is described by a constant returns to scale production function with labor augmenting productivity:

\[
Y_{jt} = AK_{jt}^{\alpha}(L_{jt})^{1-\alpha}
\]

where \( Y_{jt}, K_{jt}, L_{jt} \) denote respectively: the output level, private capital stock, and labour inputs in firm \( j \) for period \( t \); and where \( A>1 \) and \( h_j \) denote constants representing scale effects, and labor
productivity which is not firm specific, and \(0 < \alpha < 1\). Assuming perfect competition in both the goods and private factor markets, and denoting \(r_t\) and \(w_t\), the rental rate of private capital and wage rate, the first order conditions of firm \(j\)’s profit maximization are:

\[r_t = A\alpha \left( \frac{K_{j,t}}{L_{j,t}} \right)^{\alpha - 1} (h_t)^{1-\alpha}\]  \hspace{1cm} (2)

\[w_t = A(1-\alpha) \left( \frac{K_{j,t}}{L_{j,t}} \right)^{\alpha} (h_t)^{1-\alpha}\]  \hspace{1cm} (3)

According to these two conditions the marginal product of each factor of production, labour and private capital, is set equal to its price.

In line with other papers, including Kalaitzidakis and Kalyvitis (2004) or Yakita (2008), we assume that labor productivity is given by

\[h_t = \frac{K_{j,t}^\beta G_t^{1-\beta}}{L_t}\]  \hspace{1cm} (4)

where \(0 < \beta < 1\); and where \(K_t = \sum_j K_{j,t}\) stands for aggregate private capital stock, \(G_t\) the stock of public capital, and \(L_t = \sum_j L_{j,t}\) the aggregate labor input. This specification of labor productivity implies the existence of positive externalities from aggregate public and private capital, \(G_t\) and \(K_t\), onto production, with the result that the per capita labor input itself (i.e. stripped of productivity increases) is unity. This is the standard specification: Arrow (1962), Aiyagari and McGrattan (1998) and Aschauer (2000) all arrive at the same formulation.

Lastly, the first order conditions given in (2) and (3) imply that the private capital to labor ratio is the same across all firms. Therefore, in equilibrium, we have that \(\frac{K_{j,t}}{L_{j,t}} = \frac{K_t}{L_t}\) - which then allows us to rewrite (4) in aggregate terms:

\[Y_t = AK_t^\alpha (h_t L_t)^{1-\alpha} = AK_t^{\alpha + \beta(1-\alpha)} G_t^{(1-\beta)(1-\alpha)}\]  \hspace{1cm} (5)

where \(Y_t = \sum_j Y_{jt}\). Defining \(\omega = \alpha + \beta(1-\alpha)\), we rewrite the aggregate production function as

\[Y_t = AK_t^\omega G_t^{1-\omega}\]  \hspace{1cm} (6)

and the corresponding first order conditions for private capital and labor inputs, like (2) and (3),

\[r_t = A\alpha \left( \frac{K_t}{G_t} \right)^{\alpha - 1}\]  \hspace{1cm} (7)
\[ w_t = A(1 - \alpha) \left( \frac{K_t}{G_t} \right)^{\alpha} \left( \frac{G_t}{L_t} \right) \]  

(8)

In order to simplify the exposition and expressions that follow, we assume that neither private nor public capital depreciate over time. We define the public to private capital ratio to be \( X_t = G_t/K_t \).

### 2.2 Households

In this economy, young agents consume, work and have children. If they survive into old age they retire and enjoy the savings made while young. To analyse the decisions in each period, we assume the representative agent of generation \( t \) has preferences represented by the life-time utility function:

\[ \ln c_t + (1 - \lambda) \ln d_{t+1} + \varepsilon \ln n_t \]  

(9)

where \( c_t \) and \( d_{t+1} \) denote consumption in periods \( t \) and \( t+1 \); \( n_t \) the number of children; \( \varepsilon \) the priority (weight) with which having a certain number of children enters the utility function; \( \lambda \) is the probability that the individual will die before the next period; and \( \rho \in (0,1) \) is a time discount factor.

Each young agent is endowed with one unit of time for his working period (a unit per capita input). Young agents earn labor income \( w_t \) which is allocated between consumption, savings, and tax payments determined by the income tax rate \( \theta_t \). They also receive a subsidy for the child rearing time, on which the taxes are also paid. For simplicity, we assume that all taxes are paid at the same rate as income tax, and postulate the subsidy to be a constant fraction, \( \rho_n \), of the wage rate \( w_t \). The budget constraint of the young and working agents of a generation \( t \) is therefore:

\[ (1 - \theta_t)[w_t(1 - zn_t) + \rho_n w_t zn_t] = c_t + s_t \]  

(10)

where \( z > 0 \) denotes rearing time per child, \( \theta_t \) is the tax rate on labor, subsidy and capital income, and \( s_t \) denotes savings (purchase of annuity assets). Because \( 0 < \theta < 1 \) and \( w_t > 0 \), we have \((1-zn_t) > \theta^3\); and it is possible that policymakers will choose \( \rho_n \rightarrow 0 \).

As in Yaari (1965) and Blanchard (1985), we will assume the existence of an actuarially fair representative insurance company operating in the perfectly competitive market for insurance which collects the savings from private agents and invests them in private capital and/or purchases of government bonds. This company repays the returns to the agents who survive into old age. Older individuals (if alive) will therefore receive an actuarially fair payment from their purchases of

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3This is the natural assumption to make since the time spend raising children cannot exceed the unit of time which the representative worker has available to do so (section 2.2).
annuity assets $s_i$ equal to \(\frac{1+r_{t+1}}{1-\lambda} s_i\). Taking into account that income taxes have to be paid on those earnings, we can write the second period of life budget constraint as:

\[
\frac{1+(1-\theta_{t+1})r_{t+1}}{1-\lambda}s_i = d_{t+1}
\]

(11)

The problem of the young individual is then to choose consumption while working, their purchases of annuity assets/savings, and the number of children, all to maximize lifetime utility subject to the budget constraints (10) and (11). After rearranging the first order conditions obtained with respect to \(c_i, s_i\) and \(n_i\) for the young agent, we obtain the following conditions:

\[
d_{t+1} = \rho[1+(1-\theta_{t+1})r_{t+1}]
\]

(12)

and

\[
n_i = \frac{\epsilon}{(1-\theta_i)(1-\rho_w)w_tz}
\]

(13)

Using those two conditions, we obtain optimal solutions for \(s_i\) and \(n_i\):

\[
s_i = \frac{(1-\lambda)\rho}{1+(1-\lambda)\rho + \epsilon}(1-\theta_i)w_t
\]

(14)

and

\[
n_i = \frac{\epsilon}{z(1-\rho_w)[1+(1-\lambda)\rho + \epsilon]} \equiv n
\]

(15)

Two conclusions follow. First, savings are a constant fraction of after tax wage income. Second, the number of children will be constant over time, but increasing in \(\rho_w\). Indeed the last expression implies that the economy will continue to survive, \(n_t \geq 1\), so long as

\[
\epsilon \geq \frac{z(1-\rho_w)[1+(1-\lambda)\rho]}{1-z(1-\rho_w)}
\]

(16)

We assume this condition to hold, given the parameter restrictions in our model. Societies do not choose to liquidate themselves voluntarily.

### 2.3 Government

The government collects tax receipts from wages, subsidies and the returns on savings. Specifically, in each period, it levies taxes at the rate of \(\theta_i\) on wages \(w_t\), subsidies, and savings \((s_{t+1})\). It also issues public debt \(b_t\) and invests the proceeds in public capital \(G_t\). The government’s budget constraint is

\[
b_{t+1} = (1+r_t)b_t + (G_{t+1} - G_t) + \rho_w w_t z n_t N_t - \theta_i(w_t L_t + \rho_w w_t z n_t N_t + r_{t+1}s_{t+1} N_{t+1})
\]

(17)
We assume interest payments and public consumption are financed via taxes on wages, subsidies received, and savings income, whereas public debt is issued to finance public capital formation (the “golden rule” of public finance). Those assumptions imply the following two conditions hold:

\[ r_t b_t + \rho_w w_t z_n N_t = \theta_t (w_t L_t + \rho_s w_t z_n N_t) + r_t s_{t+1} N_{t+1} \]  

(18)

and

\[ b_t = G_t \]  

(19)

3 Solving for the optimal public to private capital ratio

The actuarially fair insurance company, and hence young workers, can invest in both private and public capital. Equilibrium in capital markets requires that

\[ s_t N_t = K_{t+1} + G_{t+1} \]  

(20)

We can use conditions (18) and (19), and combine them with (2) and (3), to derive an expression for the income tax rate. It is:

\[ \theta_t = 1 - \frac{1}{\alpha X + \rho_n (1-\alpha) z_n / (1-z_n) + 1} \]  

(21)

Using the equilibrium condition for capital markets (20), the solution for \( s_t \) in (14), and the fact that \( L_t = (1-z_n) N_t \) [that the labor supply from each individual equals his/her time endowment less the child rearing time], we get

\[ \frac{(1-\lambda) \rho}{1 + (1-\lambda) \rho + \varepsilon} (1-\theta_t) w_t N_t = K_{t+1} + G_{t+1} \]  

(22)

This equation determines the dynamic relationship between wages, working population, and public and private capital. Next we define the balanced growth path as a situation in which all variables grow at a constant rate. Let the aggregate growth rate be defined as

\[ \frac{G_{t+1}}{G_t} = \frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t} = \gamma_n \equiv \gamma^A \]  

(23)

where \( n \) is the constant growth rate of population. This implies that ratio of public to private capital is constant in steady state. It also implies that the tax rate \( \theta_t \) and the interest rate \( r_t \) will be constant.

Note that higher subsidy rates and taxes now lead to higher fertility, \( n \), and higher debt \( d^* \), first because \( \rho_n \) raises \( n \) by (15) and \( d^* \) by (28) below, and because it raises \( \theta_t \) by (21). But high interest rates reduce investment, as we might expect, since they raise \( \theta_t \) and lower \( K_t + G_t \) by (18) and (22).

Next we define \( \tilde{C} = \frac{(1-\lambda) \rho}{1 + (1-\lambda) \rho + \varepsilon} \), and then use (22), with (3), (18) and (21), to obtain an expression from which we can derive a closed form expression for the economy’s growth rate \( \gamma^A \):
\[
\begin{aligned}
&\left.\frac{1}{\alpha X + \rho_w (1-\alpha) zn / (1-zn) + 1}\right) \cdot A(1-\alpha) \left(\frac{K_t}{G_t}\right)^{\omega} \frac{G_t}{L_t} N_t = K_{t+1} + G_{t+1} \\
&\text{(24)}
\end{aligned}
\]

This allows us to solve for the economy's aggregate growth rate (Appendix A for the full solution):

\[
\gamma^A = \frac{A\tilde{C} \cdot \frac{1-\alpha}{1-zn}}{X^{\omega-1}(X+1)(\alpha X + C + 1)}
\]

To derive the public to private capital ratio which maximizes the aggregate growth rate along the balanced growth path we need to take the first derivative of (25) with respect to the public to private capital ratio \(X\), set equal to zero, and solve for \(X\). The result is a general solution of the form\(^4\):

\[
X_{1,2} = \frac{-\omega \left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right) \pm \sqrt{\omega^2 \left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)^2 - 4\alpha(1+\omega)(\omega-1) \left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1\right)}}{2\alpha(1+\omega)}
\]

\[
(26)
\]

With our parameter restrictions, it is easy to show that the positive solution to this equation is also positive:\(^5\)

\[
X^* = \frac{-\omega \left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right) \pm \sqrt{\omega^2 \left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)^2 - 4\alpha(1+\omega)(\omega-1) \left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1\right)}}{2\alpha(1+\omega)} > 0
\]

\[
(27)
\]

Given (6), the optimal debt to GDP ratio in this model is now:

\[
d^* = \frac{1}{A} X^{\omega}
\]

\[
(28)
\]

From (27) it follows that if the subsidy \(\rho_w \rightarrow 0\), the optimal debt level becomes independent of any demographic parameters. Otherwise, given that \(\omega < 1\), the debt burden \(d^*\) will increase with \(\rho_w\).\(^6\)

## 4 Policy implications

We have noted that, if \(\rho_w \rightarrow 0\), the optimal debt ratio becomes independent of the age-related parameters. Thus, if a government is not allowed to borrow to cover revenue shortfalls for age-related spending, the debt burden will not vary with demographic change. The policy lessons are:

\(^4\) For a detailed derivation, see again Appendix A.

\(^5\) See Appendix B. We are only interested in the positive solution since the capital stock ratio must remain nonnegative.

\(^6\) Note, if public capital becomes unproductive relative to private capital, then \((1-\omega) \rightarrow 0\) and \(X^*\) and the optimal level of debt, \(d^*\) in (28), tend to zero; a result in line with Greiner (2010) and Checherita et al. (2013).
First, what matters is not how a government chooses to spend its resources, but how it chooses to finance that spending.

Second, the subsidies described above can be interpreted in a number of different ways to cover the different types of age-related spending and/or social support in the OECD economies. The most important alternatives are:

a) Subsidies to raising children. For example, the UK provides child benefit payments which were taxed at standard rates until the 1980 and will be again from 2012. In fact any grant which is either taxed or means tested can be written as (10) under uniform grant/means test rates. Most tax codes, including that in the US, offer a tax free element per-child which, in this context, is equivalent to a net income supplement of \( \rho_w(1-\theta) wz \) taxed at the standard rate (where \( \rho_w \) is set to make this expression equal to the tax saving). That then implies (10).

b) Subsidies to education. The US, for example, taxes certain training grants and fee waivers. Subsidised loans or means tested tax reductions on fees operate the same way as child benefit, except that \( z \) now refers to time spent in education (child rearing by society rather than parents).

c) Education more generally where state spending per pupil is related to the average wage, but taxes levied at standard rates on steady state earnings fund that spending. \( z \) is now the proportion of the young population in state funded schools.

d) Health care costs. Where care givers are paid through a state subsidy; or where, as in the US, those costs contain hidden subsidies (\( \rho_w \)).

e) Sick pay. Most EU countries pay a fraction of the wage for time off sick, but tax it as income.

f) State pensions. These are taxed as income, where \( \rho_w \) is the pension replacement rate, and where \( [(1-\lambda)/n] \) replaces \( nz \) from the government budget constraint (17) onwards; \( n \) having been set in the previous generation.

Third, \( \rho_w \) itself may vary: i) debt becomes independent of age-related spending if \( \rho_w \to 0 \); ii) we might expect \( d^* \) to increase with \( \rho_w \) and reach its maximum value at \( \rho_w = 1 \). It is easy to check that is indeed the case; iii) we need to preserve the incentive to work; i.e. make the subsidy smaller than wages in work, \( w(1-zn) > \rho_w wz \). This bounds \( \rho_w \) from above: \( \rho_w < \min\{(1-zn)/zn,1\} \) which limits the debt burden as the latter increases with \( \rho_w \).

This leads to two additional conclusions: First, the optimal debt level depends, in general, on demographic parameters. However, this is not the case if governments are forbidden to borrow to cover current shortfalls for age-related/social spending. Second, these results are especially useful because they open the door to determining how a country’s best fiscal position and debt burden are affected.
(increased or decreased) by demographic changes: ageing or the baby-boomer problem (variations in $\rho$); increasing longevity, falls in the death rate, alterations in the retirement age (all variations in $\lambda$); or changes in the demand for medical benefits and increased social support or education spending (variations in $\rho_w$).

References


Appendix A: The solution to the optimal level of debt problem

We start from (22), with (21), (3) and (18), to obtain an expression from which we can derive a closed form expression for the economy’s growth rate $\gamma^A$:

$$
\hat{C} \left( \frac{1}{\alpha X + \rho w (1 - \alpha) \frac{zn}{1 - zn} + 1} \right) A(1 - \alpha) \left( \frac{K_t}{G_t} \right)^{\omega_t} \frac{G_L}{L_t} N_t = K_{t+1} + G_{t+1} 
$$

(A1)

where $\hat{C} = \frac{(1 - \lambda) \rho}{1 + (1 - \lambda) \rho + \epsilon}$. This is equation (24) in the main text. Next we set $C = \rho w (1 - \alpha) \frac{zn}{1 - zn}$ so that (A1) becomes

$$
A\hat{C} \left( \frac{1}{\alpha X + C + 1} \right)^{1 - \alpha} \frac{1}{1 - zn} X^{\omega_t} = \frac{K_{t+1}}{G_{t+1}} \frac{G_{t+1}}{G_t} + \frac{G_{t+1}}{G_t}
$$

(A2)

Using (23) we can now rewrite (A2) as

$$
A\hat{C} \left( \frac{1}{\alpha X + C + 1} \right)^{1 - \alpha} \frac{1}{1 - zn} X^{\omega_t} = \gamma^A (1 + \frac{1}{X})
$$

This allows us to solve for the economy's aggregate growth rate, $\gamma^A$, to get

$$
\gamma^A = \frac{A\hat{C} \frac{1 - \alpha}{1 - zn} X^{\omega_t - 1}}{(X + 1)(\alpha X + C + 1)}
$$

(A3)

which is the growth rate we need to optimize with respect to the public to private capital ratio.

To perform this optimisation, define $C_3 = A\hat{C} \frac{1 - \alpha}{1 - zn}$. Take the first order derivative of $\gamma^A$ (denoted by $\dot{\gamma}$) with respect to $X$:

$$
\dot{\gamma}(x) = -C_3 (\omega - 1) X^{\omega_t - 2} \left[ \alpha X^2 + (\alpha + C + 1) X + C + 1 \right] + X^{\omega_t - 1} (2\alpha X + \alpha + C + 1) \left[ X^{\omega_t - 1} (X + 1)(\alpha X + C + 1) \right]^2
$$

Setting this expression equal to zero, we can write

$$
\alpha(\omega - 1) X^{\omega_t} + (\omega - 1)(\alpha + C + 1) X^{\omega_t - 1} + (\omega - 1)(C + 1) X^{\omega_t - 2} + 2\alpha X^{\omega_t} + (\alpha + C + 1) X^{\omega_t - 1} = 0
$$

which results in a quadratic equation to be solved for the optimal public-private capital ratio $X$:

$$
\alpha(\omega + 1) X^2 + \omega(\alpha + C + 1) X + (\omega - 1)(C + 1) = 0
$$

The required solution is therefore:

$$
X_{1,2} = \frac{-\omega \left( \rho_w \frac{(1 - \alpha) zn}{1 - zn} + 1 + \alpha \right) \pm \sqrt{\omega^2 \left( \rho_w \frac{(1 - \alpha) zn}{1 - zn} + 1 + \alpha \right)^2 - 4\alpha(1 + \omega)(\omega - 1) \left( \rho_w \frac{(1 - \alpha) zn}{1 - zn} + 1 \right)}}{2\alpha(1 + \omega)}
$$

which is (26) in the main text.
Appendix B: The positive root

We now demonstrate that our solution for $X^*$ is in fact positive. We start from (27) in the text. Since $\omega > 0,0 < z < 1, n \geq 1, 0 < \alpha < 1$, it follows that $\omega \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right) > 0$ and $2\alpha(1+\omega) > 0$.

Therefore to get a positive solution, $X^* > 0$, we need the numerator of (27) to be positive. It will be positive if and only if

$$-\omega \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right) + \sqrt{\omega^2 \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right)^2 - 4\alpha(1+\omega)(\omega-1) \left( \frac{(1-\alpha)zn}{1-zn} + 1 \right) > 0 \right).$$

We therefore need to check if this condition is in fact satisfied for the parameter constraints that we have imposed. We require

$$\sqrt{\omega^2 \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right)^2 - 4\alpha(1+\omega)(\omega-1) \left( \frac{(1-\alpha)zn}{1-zn} + 1 \right) \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right)} > \omega \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right);$$

or

$$\omega^2 \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right)^2 - 4\alpha(1+\omega)(\omega-1) \left( \frac{(1-\alpha)zn}{1-zn} + 1 \right) \left( \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha \right)^2 > 0;$$

or

$$-4\alpha(1+\omega)(\omega-1) \left( \frac{(1-\alpha)zn}{1-zn} + 1 \right) > 0$$

which is true since $-4\alpha(1+\omega)(\omega-1) = -4\alpha(\omega^2 - 1) > 0$ and $\frac{(1-\alpha)zn}{1-zn} + 1 > 0$ necessarily hold.

Hence $X^*$ in (27) is positive.