A Model of Two-stage Electoral Competition

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Abstract. The paper presents a simple model of two-stage electoral competition with two parties: an incumbent party and a challenging party, and shows that the presence of a primary election causes a more moderate policy to be selected in equilibrium and increases the probability that a candidate with higher personal qualities, called valence, wins in a general election. The model resembles the US presidential election and many analogues are observed in the world, on which developing a theoretical model has largely escaped attention until recently. The model is analytically tractable, and provides a vehicle for answering normative questions about holding primary elections. Finally, our findings provide empirical predictions on primaries and the roles of valences in election.

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1 Introduction

In this paper, we develop a theoretical framework of a primary and a general election. The main objective is to further our understanding of what the existence of primaries would bring in to our society. Despite of many real examples across the world, the theoretical works on party primaries are quite scarce in the literature. We develop an approach to the study of democratic policy-making where voters judge their candidates in two dimensions: non-policy personal qualities referred to as valence and policy promises, while politicians foresee the events of winning both in a primary and a general election.

Our approach has a number of attractive features. First, the model is featured with properties that are observed in reality. Especially, our model, although it is not confined to, resembles the US primaries. Second, the model is analytically tractable having a unique equilibrium, being able to handle multidimensional issue between valence and policy promises very naturally. Third, it provides a vehicle for answering normative questions about holding a primary election. Fourth, our findings are consistent with the empirical research on primaries and the roles of valences in election.

In American history, George Washington is the only president that was elected as the unanimous choice of the electors up until now. Since then, political conditions gave rise to the current style of the primary system through a number of reforms. In practice, primary elections are commonly used to select a party representative who competes with other parties’ nominees in the subsequent general election in many other countries than the US, including some European countries, and South American countries. In 2009, the UK Conservative Party experimented with the use of open primaries to select two of its parliamentary candidates. Political parties throughout Latin America rely increasingly on primary elections to select presidential candidates. An increasing number of empirical studies have considered the effects of primaries using more recent or international data sets. By using data from every democratic presidential election in Latin America since the late 1970s, Carey and Polga-Hecimovich (2006) find evidence of a primary bonus, that is, holding other things equal, primary selected candidates are stronger.

\begin{footnote}
1Ware (2002) studies the origins of direct primary elections in the US since the 1920s. It claims that the direct primary was the result of an attempt by politicians to subject their previously informal procedures to formal rules, that started in the late 1880's. Because it was impossible to make effective changes to the caucus-convention system of selecting election candidates, politicians turned to the direct primary. To study how primaries (or caucuses) have shown up historically, the roles of political parties might be important to be considered. Chhibber and Kollman (2009) study the origins of political parties in Canada, Great Britain, India and the US. They state that the earliest party caucuses by Hamilton and Jefferson in the 1790s were created to organize the House of Representatives into policy-making blocs. They were then used to choose presidential candidates (See page 86 in Chhibber and Kollman, 2009)
\end{footnote}
than those selected by other procedures. By using samples of Democratic caucus attenders and state convention delegates from Iowa in 1984, Stone et al. (1992) find that nomination choice cannot be explained by relying on simple candidate preferences, and it rather emerges from “strategic interaction” between several factors such as winning chances and candidate factors.

However, in the theoretical literature, most models of electoral competition assume that candidates representing their political parties compete in one-stage elections. Recently, within the framework proposed by Coleman (1971), Owen and Grofman (2006) develop a model of two-stage electoral competition in two-party contests in one dimension and study candidate’s policy position. Adams and Merrill (2008) develop a model to postulate that primary elections may allow a party to identify a high-quality nominee. Then, they show the existence of a Nash equilibrium and numerically demonstrate the equilibrium configuration when both or one party holds a primary. Recently, Serra (2011) also proposes a model of party elites such that party leaders decide whether or not to hold a primary election. Serra (2011) solves a sub-game perfect equilibrium by assuming that each party has one candidate and tries to maximize their party’s median voter, while the party elites would choose to use their endorsement rather than a primary if his endorsed candidate gives a better utility for them. The focus of Serra (2011) is the parties’ choice of a candidate, while the focus of Adams and Merrill (2008) is the candidates’ choice of platforms. In this paper, we characterize the interaction between these two forces.

Our model follows the strand of two-party competition as in Downs (1957).² McGann (2002) shows the policy divergence in unidimensional two-party competition and predicts that the policy promises of the candidates chosen by the primaries in each party will be somewhere between the party median and the overall median (see also Cooper and Munger, 2000). In the early 1970s, a similar prediction has been made in the two streams of models, namely Aranson and Ordeshook (1972) and Coleman (1971, 1972).

We adopt the ideas in Owen and Grofman (2006), and extend the analysis to a two-dimensional case where voters judge candidates on two aspects: policy promises and valence. To model the median voter’s bliss point, we adopt the idea from Roemer (1997), which assumes that parties are uncertain about the distribution of traits among the voters who will turn up in the general election and thus there is uncertainty about the realization of the median voter’s position. We then consider an equilibrium such that politicians maximize their winning probability to the incumbent candidate in a general election subject to the constraint such that they do their best to win a primary election.

In our model, as politicians are committed to one policy for both elections, a policy promise that each candidate proposes must satisfy a certain consistency between the two stages. Technically, we impose a constraint for politicians to do their best to defeat the other candidate in terms of their party’s median voter’s expected utility and in this way, we propose a concept

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²Grofman (2004) provides a review on the insights of Downs (1957) into two-party competition
of political equilibrium with primaries. Models of party competition that are rested on Downs (1957) have recognized that there are centrifugal and centripetal forces in party competition. In our model, a centripetal force is an appeal to the overall median voter in a general election and a centrifugal force is an appeal to their own party’s median voter in a primary. Because our equilibrium concept explicitly includes both of the two forces, these two forces drives the policy divergence in the two-dimensional party competition and the distance of each candidate’s policy promise depends on the difference of valences and the distribution of the overall median voter in equilibrium.

Our findings in the paper include the followings. First, we show that the existence of the primary election makes a more moderate policy to be selected in equilibrium and at the same time, it increases the probability that a candidate with a higher personal quality that all the voters commonly appreciate wins in a general election. Second, we analyze an interaction between the salience of valence and platforms in equilibrium. Finally, we show how the distribution of the median voter’s bliss point in the general election affects the policy decision for each candidate and characterize the relationship between policy promises and valences in equilibrium.

There is the large body of empirical work on valences and policy promises in the literature. Among them, Hirano and Snyder (2013) argue that the literature underestimates the value of primaries, and primary elections are most needed in safe constituencies, where the advantaged party’s candidate can usually win the general election. Many empirical studies have analyzed how party elites’ valences affect voters’ electoral supports for parties (McCurley and Mondak, 1995, Mondak, 1995, Stone and Simas, 2010), although the relationship between valences, policy decisions and winning chances remain unclear and uncertain. Scholarly evidence from studies of the U. S., and from recent cross-national research of Western Europe, suggests that changes to parties’ or candidates’ valences carry important electoral consequences (see Abney et al., 2013). On the one hand, some empirical works find that as parties’ ideological distance becomes smaller, valence becomes more important. Clark and Leiter (2013) bring together research on valence and party dispersion to examine whether the ideological dispersion of parties in a party system mediates the electoral impact of valence by using the data from nine Western European countries over the period 1976 to 2003. In their study of recent British elections, Green and Hobolt (2008) find that as parties have converged ideologically, valence-based evaluations of parties have become more important to voters than ideological positions. Similarly, in their research on elite depolarization in Britain, Adams et al. (2012) conclude that voters forego policy-based voting when the policy dimension of party competition is less salient, and this situation is likely to occur when parties are less ideologically dispersed on the policy dimension. On the other hand, in a slightly different setting, it is found that the relationship between valence and vote choice weakens as the ideological distance between the candidates grows. In their study of U.S. House elections, Buttice and Stone (2012) examine the
relationship between candidates’ valence and the ideological distance between the candidates. Analyses of their probabilistic voting models reveals that Republican candidate’s valence has a reduced effect on his or her vote share in their district as the ideological difference between the candidates increases in magnitude.

Moreover, despite of the large empirical body of researches, the literature does not yet produce candid predictions on the effects of policy promises and valences to voters. It suggests that we need a micro-founded tractable model of two-stage election with a coherent equilibrium concept to understand the consequences of primaries and the effects of the salience for valences on policy promises. This paper provides such a framework and provides a testable hypothesis on valences and policy promises in the presence of primary elections. In summary, we obtain the following predictions from our model:

- as the incumbent’s valence becomes higher,
  - the challenging candidate chooses a policy promise that is more favorable to their own party;
  - the challenging party’s winning chances decrease;

- as the ideological difference decreases, valence becomes more important on electoral outcome.

The remainder of the paper is organized as follows. Section 2 details the model. Section 3 undertakes the equilibrium analysis of policy positions by candidates. The final section concludes.

2 The Model

In the model, we assume that there are two parties: an incumbent party and a challenging party. The incumbent party has one politician, who we call the “incumbent.” The challenging party has two candidates. We assume that there are two stages of election: a primary and a general election. The two candidates in the challenging party are assumed to compete in a primary election. Then the winner in a primary election competes with the incumbent in a general election and the final winner in a general election obtains the office. Each politician is characterized by two aspects: policy promise and valence. A valence variable benefits all voters independently of their political opinions. Examples of valence include integrity, educational background and a “clean” past. We assume that a policy space is $[0, 1]$.

There is a continuum of voters distributed on the policy space with mass 1. Voter $i$ is described by his or her preferred policy $i \in [0, 1]$. Let $j$ denote the candidate who wins the general election, and whose policy promise is $p_j \in [0, 1]$. We assume that politicians in
the challenging party choose one policy promise that would be common in a primary and a
general election if each wins the candidacy in the primary election. Let $v_j \in R$ be a valence
of politician $j$ and $| \cdot |$ denote the Euclidean distance. As each voter is associated with its bliss
point, we call each voter its bliss point. The utility of voter $i$ is:

$$u_i(v_j, p_j) = h(v_j) - g(|p_j - i|),$$  \hspace{1cm} (1)$$

for a strictly increasing function $h : R \rightarrow R$ and a continuously differentiable and increasing
concave function $g : [0, 1] \rightarrow R_+$. For simplicity of calculation, we assume $g(0) = 0$.

We assume that a certain portion of voters vote for a primary election in the challenging
party. Suppose that $\delta$ is a set of voters who register for the challenging party’s primary election.
We define a median voter’s bliss point in the challenging party by $R$ as follows:

$$|\{r \in \delta : r \leq R\}| \geq \frac{|\delta|}{2} \quad \text{and} \quad |\{r \in \delta : r \geq R\}| \geq \frac{|\delta|}{2}.$$  

Assume that $R$ is exogenously given and the incumbent’s policy promise is $\bar{p}$. We assume
that due to commitment to the previous policy and her credential constraint, $\bar{p}$ is fixed and
publicly known. Without loss of generality, we assume $\bar{p} < R$. The incumbent’s valence is
denoted by $\bar{v} \in R$. Two potential candidates in the challenging party, called challenger $L$
and $H$, have valences, $v_L$ and $v_H$ with $v_L < v_H$, respectively. We assume that $\bar{v}$, $v_L$ and $v_H$
are public information. We can also interpret $h(\bar{v})$ which explicitly also includes incumbency
advantage. It is reasonable to assume that there are reasons for incumbency advantage such as
name recognition and access to various perquisites of office. Or, if an incumbent has gotten
involved in a scandal prior to the election, this disadvantage can be expressed as a lower value
of $h(\bar{v})$.

As there is a median voter in the challenging party, we explicitly call the median voter
for the whole population the “overall median voter.” We denote the bliss point of the overall
median voter by $m$. We assume that $m$ is randomly distributed in $[\bar{p}, R]$, whose cumulative
distribution function is given by a function $F$. Suppose that $F$ is twice differentiable and the
hazard rate $F'(x) \frac{1}{1-F(x)}$ is increasing in the region. A set of political party members are included
in a set of voters. We assume that while the distribution $F$ is a common knowledge in the
model, the distribution of the voters who actually turn up at the election is unknown, causing
the uncertainty about which of the two parties will win the general election.

All voters equally benefit from valence but have different preferences regarding policy,
which is represented by a single-peaked preference as in (1). The median voter theorem holds

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3 Morton and Williams (2001) and some other works show that voters update their beliefs in response to
campaign events and learn about the candidates during the campaign. This provides support for this assumption
that the valences are public information.

4 This assumption simplifies our analysis. We can relax this assumption by comparing the outcome in a point
that is not twice differentiable with candidate equilibrium outcomes to solve for an equilibrium. Later, we provide
an example that when this property fails, how our idea works.
in this policy space. The voting behavior in the general election is simple. Since there are only two competing policies, each voter votes for the candidate whose policy and valence yield higher utility for them.

A challenger is denoted by \( k \in \{H, L\} \) and the other challenger is denoted by \(-k \) (e.g., when \( k = L, -k = H \)). Before the elections, challenger \( H \) and \( L \) choose policy promises, \( p_L \) and \( p_H \), respectively. Let \( \pi : \mathbb{R} \times [0, 1] \rightarrow [0, 1] \) denote a probability that candidate \( k \) for each \( k \in \{H, L\} \) beats the incumbent in a general election. Then by the median voter theorem, for each \( k \in \{L, H\} \), \( \pi \) is indeed the probability that the overall median voter \( M \) (that is a realization of \( m \) votes) for challenger \( k \):

\[
\pi(v_k, p_k) = \Pr [u_M(v_k) \geq u_M(\bar{v}, \bar{p})].
\] (2)

We now compute a winning probability in a general election for each challenger. Let \( M \) denote a realized overall median voter’s bliss point. Then, challenger \( k \) wins if when \( M - p_k \geq 0 \),

\[
h(\bar{v}) - h(v_k) \leq g(M - \bar{p}) - g(M - p_k)
\]
or when \( M - p_k < 0 \),

\[
h(\bar{v}) - h(v_k) \leq g(M - \bar{p}) - g(p_k - M).
\]

Define \( d(v_k) \) for each \( k \in \{H, L\} \) by

\[
d(v_k) = \begin{cases} 
  g^{-1}(h(\bar{v}) - h(v_k)) + \bar{p} & \text{if } h(\bar{v}) \geq h(v_k); \\
  -g^{-1}(h(v_k) - h(\bar{v})) + \bar{p} & \text{if } h(\bar{v}) < h(v_k).
\end{cases}
\]

For each \( k \in \{H, L\} \), define a function \( Y_k : [\bar{p}, 1] \rightarrow \mathbb{R} \) as follows. For each \( p_k \in [\bar{p}, 1] \), \( Y_k(p_k) \) satisfies

\[
h(\bar{v}) - h(v_k) = g(Y_k(p_k) - \bar{p}) - g(p_k - Y_k(p_k)),
\]

if there is a \( Y_k(p_k) \in [\bar{p}, p_k] \), and otherwise, \( Y_k(p_k) = p_k \).

In computing a winning probability, \( Y_k(p_k) \) becomes important for each \( p_k \in [\bar{p}, R] \).

**Proposition 1.** For each challenger \( k \in \{H, L\} \) with policy promise \( p_k \in [0, 1] \), the probability of winning the general elections is

\[
\pi(v_k, p_k) = \begin{cases} 
  0 & \text{if } p_k < d(v_k) \\
  1 & \text{if } d(v_k) \leq p_k < \max \{d(v_k), Y_k^{-1}(\bar{p})\} \\
  1 - F(Y_k(p_k)) & \text{if } \max \{d(v_k), Y_k^{-1}(\bar{p})\} \leq p_k < Y_k^{-1}(R) \\
  0 & \text{if } p_k \geq Y_k^{-1}(R)
\end{cases}
\]

where the interval \([d(v_k), \max \{d(v_k), Y_k^{-1}(\bar{p})\}]\) is non-empty if and only if \( h(v_k) \geq h(\bar{v}) \).
For each \( k \in \{L, H\} \), when challenger \( k \) wins a primary, voter \( r \)'s expected utility is given by
\[
E\Pi_r(v_k, p_k) = (1 - \pi(v_k, p_k))u_r(\bar{v}, \bar{p}) + \pi(v_k, p_k)u_r(v_k, p_k).
\]

(3)

In a primary election, we assume that each voter votes for a politician to maximize her expected utility. We assume that \( \bar{p} < r \) for each \( r \in \delta \). This means that voters registering for a primary have bliss points that are more rightish. Also, we assume that
\[
h(\bar{v}) - h(v_L) < g(R - \bar{p}).
\]

(4)

This assumption sets an upper bound on the difference in valences between the incumbent and challengers. This assumption implies that if challenger \( L \) chooses a reasonable policy for voters in the challenging party, median voter \( R \) prefers him to the incumbent and the policy promises also matter over valences. Let \( \bar{r} \) satisfy \( u_r(v_L, R) = u_r(\bar{v}, \bar{p}). \) We assume that for all \( r \in \delta, r \geq \bar{r} \).

**Lemma 1.** \( u_r(v_k, R) > u_r(\bar{v}, \bar{p}) \) for all \( r \in (\bar{r}, R) \) for each \( k \in \{H, L\} \).

**Proof.** As \( u_r(v_H, R) > u_r(v_L, R) \), obviously \( u_r(v_H, R) > u_r(\bar{v}, \bar{p}) \). Fix \( k \in \{H, L\} \) and then for all \( R \geq r > \bar{r} \), \( u_r(v_k, R) > u_r(\bar{v}, \bar{p}) \), because
\[
u_r(v_k, R) = h(v_k) - g(R - \bar{r}) \geq u_r(\bar{v}, \bar{p}) = h(\bar{v}) - g(\bar{r} - \bar{p})
\]

(5)

and
\[
u_r(v_k, R) = h(v_k) - g(R - r) > u_r(\bar{v}, \bar{p}) = h(\bar{v}) - g(r - \bar{p}).
\]

(6)

We now show that in a primary election, we can also focus on the median voter in the challenging party.\(^5\) A policy \( p_k^e \) is called a **Condorcet winner** for challenger \( k \in \{H, L\} \) if for any \( p_k \in [0, 1] \),
\[
|\{r \in \delta : E\Pi_r(v_k, p_k^e) \geq E\Pi_r(v_k, p_k)\}| \geq \frac{|\delta|}{2}.
\]

**Proposition 2.** Let \( r \in [R, 1] \). Then, voter \( r \)'s expected payoffs are single-peaked at \( \min\{R, p_k^e\} \) in the interval \([d(v_k), R]\) for each \( k \in \{H, L\} \) where \( g'(r - p_k^e) = \frac{\pi'(v_k, p_k^e)(u_r(v_k, p_k^e) - u_r(\bar{v}, \bar{p}))}{\pi(v_k, p_k^e)}. \)

**Proof.** Let \( r \in [R, 1] \) and \( p_k \leq r \). Then,
\[
E\Pi_r(v_k, p_k) = (1 - \pi(v_k, p_k))(h(\bar{v}) - g(r - \bar{p})) + \pi(v_k, p_k)(h(v_k) - g(r - p_k)).
\]

Notice that \( E\Pi_r(v_k, d(v_k)) = h(v_k) - g(r - d(v_k)) \) and obviously \( E\Pi_r(v_k, d(v_k)) < E\Pi_r(v_k, r). \)

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\(^5\)Chen (2009) proves a similar result in one-dimensional competition.
Take the first derivative and then
\[ E\Pi'_r(v_k, p_k) = \pi'(v_k, p_k)(u_r(v_k, p_k) - u_r(\bar{v}, \bar{p})) + \pi(v_k, p_k)g'(r - p_k). \]

Then, let
\[ g'(r - p_k) = -\frac{\pi'(v_k, p_k^r)(u_r(v_k, p_k^r) - u_r(\bar{v}, \bar{p}))}{\pi(v_k, p_k^r)}, \]

where
\[ \frac{\pi'(v_k, p_k^r)}{\pi(v_k, p_k^r)} = \frac{Y_k^r(p_k^r)F'(Y_k(p_k^r))}{1 - F(Y_k(p_k^r))}. \]

(7)

Obviously, \( u_r(v_k, p_k^r) > u_r(\bar{v}, \bar{p}) \). Thus, when \( (\log \pi(v_k, p_k^r))' \) is decreasing, we obtain the desired result.

\[ \square \]

**Proposition 3.** Let \( r \in [\bar{r}, R] \). Voter \( r \)'s expected payoff is a \( U \)-shaped and takes minimum at
\[ \max\{p_k, Y_k^{-1}(R)\} \]
where \( g'(\bar{p}_k^r - r) = \frac{\pi'(v_k, \bar{p}_k^r)(u_r(v_k, \bar{p}_k^r) - u_r(\bar{v}, \bar{p}))}{\pi(v_k, \bar{p}_k^r)} \) in the interval \([p_k, Y_k^{-1}(R)]\) for some \( p_k \geq r \).

**Proof.** As \( r \leq p_k \),
\[ E\Pi_r(v_k, p_k) = (1 - \pi(v_k, p_k))(h(\bar{v}) - g(r - \bar{p})) + \pi(v_k, p_k)(h(v_k) - g(p_k - r)), \]
and especially when \( r = p_k \),
\[ E\Pi_r(v_k, p_k) = (1 - \pi(v_k, p_k))(h(\bar{v}) - g(r - \bar{p})) + \pi(v_k, p_k)(h(v_k)). \]

Notice that
\[ E\Pi_r(v_k, Y_k^{-1}(R)) = h(\bar{v}) - (r - \bar{p}), \]
and
\[ E\Pi_r(v_k, R) = (1 - \pi(v_k, p_k))(h(\bar{v}) - (r - \bar{p})) + \pi(v_k, p_k)(h(v_k) - g(R - r)). \]

By Lemma 1, it is clear that \( E\Pi_r(v_k, r) > E\Pi_r(v_k, R) > E\Pi_r(v_k, Y_k^{-1}(R)) \)

Take the first the first derivative of the expected preference, and then
\[ E\Pi'_r(v_k, p_k) = \pi'(v_k, p_k)(u_r(v_k, p_k) - u_r(\bar{v}, \bar{p})) - \pi(v_k, p_k)g'(p_k - r). \]

If there is no policy \( p'_k \) to satisfy \( u_r(v_k, p'_k) = u_r(\bar{v}, \bar{p}) \), \( u_r(v_k, p_k) - u_r(\bar{v}, \bar{p}) > 0 \)
and the first order condition is always negative, which indicates that the expected payoff is strictly decreasing in \([p_k, Y_k^{-1}(R)]\). If there is such a \( p'_k \), let
\[ g'(\bar{p}_k^r - r) = \frac{\pi'(v_k, \bar{p}_k^r)(u_r(v_k, \bar{p}_k^r) - u_r(\bar{v}, \bar{p}))}{\pi(v_k, \bar{p}_k^r)}. \]

As \( g' \) is strictly positive and \( \pi' \) is strictly negative, we must have \( \bar{p}_k^r > p'_k \). The first order condition is strictly negative in \([p_k, \bar{p}_k^r]\). Consider \( p_k > \bar{p}_k^r \). Then, \( u_r(v_k, p_k) - u_r(\bar{v}, \bar{p}) < 0 \).

As \( E\Pi_r(v_k, p_k) < E\Pi_r(v_k, R) \), we obtain a desired result. \( \square \)
We particularly denote $p_k^R$ by $\hat{p}_k$ for each $k \in \{H, L\}$.

**Proposition 4.** For each candidate $k \in \{H, L\}$, let $R_k^* = \min\{\hat{p}_k, R\}$, and then median voter $R$’s optimal policy $R_k^*$ is the Condorcet winner in a primary.

**Proof.** Fix $k \in \{H, L\}$.

(Case I) $\hat{p}_k \leq R$. Suppose that $p_k < R_k^* = \hat{p}_k$. Consider voter $r$ with $r > R$. As shown in Proposition 2, voter $r$’s expected payoff exhibits a single-peakedness. Thus,

$$\left\{ r \in \delta : \mathbb{E}\Pi_r(v_k, \hat{p}_R) \geq \mathbb{E}\Pi_r(v_k, p_k) \right\} \geq \frac{|\delta|}{2}.$$

Similarly, we can prove that $R_k^*$ beats any policy $p_k \geq \hat{p}_R$, because voters whose bliss points are smaller than $R$ strictly prefer $R_k^*$.

(Case II) $\hat{p}_k > R$. Suppose that $p_k \leq R_k^* = R$. As voter $r$ with $r > R$ strictly prefers $R$, $R_k^*$ beats any policy $p_k$. Now, suppose that $p_k > R_k^*$. Consider voter $r$ with $r < R$. Then, by Lemma 1, $u_r(v_k, R) > u_r(\bar{v}, \bar{p})$. As $\mathbb{E}\Pi_r(\bar{v}, 2R - d(v_k)) = u_r(\bar{v}, \bar{p})$, $\mathbb{E}\Pi_r(v_k, R) > \mathbb{E}\Pi_r(\bar{v}, 2R - d(v_k))$. Thus, voter $r$ strictly prefers $R$. Therefore, $R$ strictly beats $p_k$. \(\square\)

To define an equilibrium for the two-stage game, we start with the competition in the first stage. Imagine the process of each candidate’s proposing a policy promise sequentially. Given $p_{-k}$, Challenger $k$ chooses $p_k$ to do as good as the other challenger and thus

$$\mathbb{E}\Pi_R(v_k, p_k) \geq \mathbb{E}\Pi_R(v_{-k}, p_{-k}). \quad (8)$$

When (8) holds, we can say that median voter $R$ is at least indifferent about challenger $k$ compared to challenger $-k$. We say that $p_k$ weakly beats $p_{-k}$ if (8) is satisfied. Define the set of challenger $k$’s promises that weakly beat $p_{-k}$ as follows:

$$\mathcal{U}_k(p_{-k}) = \{ p \in [0,1] : \mathbb{E}\Pi_R(v_k, p) \geq \mathbb{E}\Pi_R(v_{-k}, p_{-k}) \}.$$

In the end of the agenda proposing process, one challenger would run out of his choice. Suppose that challenger $k$ runs out of his proposal to beat challenger $-k$. Then, his last choice would be a maximizer of $\mathbb{E}\Pi_R(v_k, p_k)$. Instead of explicitly modelling this process, we use a reduced form resulted from this process as a constraint for their optimization problem. Because each challenger has to choose a policy promise that would be his promise for both a primary and a general election, he does just as good as to beat the other candidate and his main goal is to maximize the winning probability in the general election. If he can do better than the other candidate, he would choose a promise that would beat the other candidate. Hence, his promise $p_k$ should satisfy (8) if there is such a $p_k$, and at the end of the process, there is no $p$ to satisfy $\mathbb{E}\Pi_R(v_k, p) > \mathbb{E}\Pi_R(v_{-k}, p_{-k})$ for at least one $k \in \{H, L\}$. Then, under this circumstance, his last choice $p_K$ must maximize $\mathbb{E}\Pi_R(v_k, p_k)$.  

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Lastly, we define a *tie-breaking rule* by assuming that if $E\Pi_R(v_k, p_k) = E\Pi_R(v_{-k}, p_{-k})$, challenger $k$ wins if and only if for an equilibrium policy promise $p_{-k}$, there is a $p'_k$ such that $E\Pi_k(v_k, p'_k) > E\Pi_{-k}(v_{-k}, p_{-k})$. If none of the challengers has or both of the challengers have such a policy, then a winner in a primary is randomly selected.

In summary, the timing of the game is given as follows:

**Stage 1** In the beginning of the game, the challenging candidates simultaneously choose their policy promises. A primary election is held for the challenging party.

**Stage 2** A general election is held. A winning candidate’s policy promise is implemented. The payoffs are then made to voters.

Now we define a political equilibrium in our model.

**Definition 1.** A political equilibrium in a game of two-stage electoral competition consists of policy promises $(p^*_L, p^*_H)$ such that for each $k \in \{L, H\}$, $p^*_k$ solves

$$\max_{p_k} \pi(v_k, p_k),$$

subject to the constraints such that for each $k \in \{H, L\}$,

1. if $U_k(p^*_k) \neq \emptyset$, then $p^*_k \in U_k(p^*_k)$;

and for at least one $k \in \{H, L\}$ and $-k \in \{H, L\}$,

2. $E\Pi_R(v_k, p^*_k) \geq E\Pi_R(v_{-k}, p_{-k})$ for all $p_{-k} \in [0, 1]$; and

3. $E\Pi_R(v_{-k}, p^*_k) \geq E\Pi_R(v_{-k}, p_{-k})$ for all $p_{-k} \in [0, 1]$.

In our model, candidates try to maximize the winning probability to the incumbent subject to the constraints for a primary.\(^6\)

### 2.1 An Equilibrium with a Primary

In this section, we solve for the equilibrium in the game of two-stage election. First, we compute the policy promises of challenger $k$ that maximize the expected payoff for the median

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\(^6\)Wittman (1973, 1977) and many subsequent works modified the standard model to allow candidates to care about winning and policy. Models of strategic voting such as Black (1978) or Cain (1978) allow voters to care about candidate’s chances of winning. In recent years, Prat (2002) or Takayama (2007) models political parties to maximize their winning probability. In relation to the constraints for a primary, by using samples of Democratic caucus attenders and state convention delegates from Iowa in 1984, Stone *et al.* (1992) finds that an expected utility model that includes both preferences and candidate chances of winning most successfully predicts candidate choices in both samples.
voter \( R \) in the challenging party. In this way, we study what policy promises would satisfy the constraints for our equilibrium.

Let \( I_k = [d(v_k), Y_k^{-1}(R)] \) and let \( I_k^o \) denote an interior set of \( I_k \) for each \( k \in \{H, L\} \). By the definition of \( Y_k^{-1} \) for each \( k \in \{H, L\} \), if there is a \( Y_k^{-1}(R) \in [R, 1] \) to satisfy

\[
h(\bar{v}) - h(v_k) = g(R - \bar{p}) - g(Y_k^{-1}(R) - R),
\]

then \( Y_k^{-1}(R) \) satisfies (10) and \( R < Y_k^{-1}(R) \) (by Assumption 4, the equality does not hold). If there is not such a \( Y_k^{-1}(R) \), then \( Y_k^{-1}(R) = R \). Thus, we obtain \( R \leq Y_k^{-1}(R) \) by definition of \( Y_k^{-1} \). Thus we obtain \( R \leq Y_k^{-1}(R) \) for each \( k \in \{H, L\} \). Also, as \( R \in I_k \) for each \( k \in \{H, L\} \), \( I_k \neq \emptyset \) for each \( k \in \{H, L\} \).

**Proposition 5.** In equilibrium, \( p_k^* \in I_k \) for each \( k \in \{H, L\} \).

**Proof.** Fix \( k \in \{H, L\} \). Suppose not and \( p_k^* \in [0, 1] \setminus I_k \). Then,

\[
\Pi_R(v_k, p_k^*) = u_R(\bar{v}, \bar{p}). \tag{11}
\]

Then, Assumption (4) indicates that

\[
\Pi_R(v_k, p_k^*) = u_R(\bar{v}, \bar{p}) < u_R(v_k, R). \tag{12}
\]

As \( u_R(\bar{v}, \bar{p}) < u_R(v_{-k}, p_{-k}) \) for all \( p_{-k} \in I_{-k} \), the equilibrium condition (1) indicates that

\[
\Pi_R(v_{-k}, p_{-k}^*) \geq \Pi_R(v_k, p_k^*). \tag{13}
\]

Then, the equilibrium condition (2) implies that for any \( p_k \in I_k \),

\[
\Pi_R(v_{-k}, p_{-k}^*) \geq \Pi_R(v_k, p_k). \tag{14}
\]

However, (12) contradicts the equilibrium condition (3), because \( R \in I_k \) and (12) imply \( \Pi_R(v_k, p_k^*) < \Pi_R(v_k, R) \).

Proposition 5 implies that in equilibrium, challenger \( k \) chooses a policy promise from \( I_k \), because otherwise a winning probability in a general election is zero for any \( p_k \not\in I_k \). So, we can restrict our attention to policy promises in \( I_k \) for each \( k \in \{H, L\} \). Let \( \hat{p}_k \in \arg\max_{p \in [\bar{p}, R]} \Pi_R(v_k, p) \) for each \( k \in \{H, L\} \).

**Lemma 2.** It is held that

(a) \( \Pi_R(v_H, \hat{p}_H) > \Pi_R(v_L, \hat{p}_L) \);

(b) \( \Pi_R(v_H, p_H^*) \geq \Pi_R(v_L, p_L) \) for any \( p_L \in I_L \).
Proof. Notice that as \( h(\bar{v}) - h(v_L) > h(\bar{v}) - h(v_H) \), monotonicity of \( g \) implies that \( Y_L^{-1}(R) < Y_H^{-1}(R) \) in (10). As \( d(v_L) > d(v_H) \), \( I_L \subset I_H \), which implies \( \hat{p}_L \in I_H \). By the definition of \( \pi \), we know that \( \pi(v_H, \hat{p}_L) > \pi(v_H, \hat{p}_L) \) as \( h(v_H) > h(v_L) \). Then, as \( \hat{p}_L \) maximizes \( E\Pi_R(v_L, \hat{p}_L) \), \( u_R(v_L, \hat{p}_L) > u_R(\bar{v}, \bar{p}) \) (otherwise, it would contradict the maximality because \( u_R(\bar{v}, \bar{p}) < u_R(v_L, R) \) by Assumption 4 and continuity of \( u_R \) in \( I_L \) implies the existence of \( p \) with \( u_R(v_L, p) > u_R(\bar{v}, \bar{p}) \)). Then, we obtain \( u_R(\bar{v}, \bar{p}) < u_R(v_L, \hat{p}_L) < u_R(\bar{v}, \bar{p}) \), and thus

\[
E\Pi_R(v_H, \hat{p}_L) \geq (1 - \pi(v_H, \hat{p}_L))u_R(\bar{v}, \bar{p}) + \pi(v_H, \hat{p}_L)u_R(v_H, \hat{p}_L) > (1 - \pi(v_H, \hat{p}_L))u_R(\bar{v}, \bar{p}) + \pi(v_H, \hat{p}_L)u_R(v_H, \hat{p}_L) > (1 - \pi(v_L, \hat{p}_L))u_R(\bar{v}, \bar{p}) + \pi(v_L, \hat{p}_L)u_R(v_L, \hat{p}_L) = E\Pi_R(v_L, \hat{p}_L).
\]

(15)

Because \( E\Pi_R(v_H, \hat{p}_L) \geq E\Pi_R(v_H, \hat{p}_L) \), we obtain (a). To prove (b), suppose that \( E\Pi_R(v_H, p^*_H) < E\Pi_R(v_L, p^*_L) \) for some \( p_L \in I_L \). As \( U_L(p_H^*) \neq \emptyset \), \( E\Pi_R(v_H, p^*_H) \leq E\Pi_R(v_L, p^*_L) \). By (a), as \( U_H(p_L^*) \neq \emptyset \), \( E\Pi_R(v_H, p^*_H) \geq E\Pi_R(v_L, p^*_L) \). Combining them, we obtain

\[
E\Pi_R(v_H, p^*_H) = E\Pi_R(v_L, p^*_L),
\]

which contradicts with the equilibrium condition (2) and (3).

By using Lemma 2 and a tie-breaking rule, we obtain the following proposition.

**Proposition 6.** In equilibrium, the following holds:

(I) \( E\Pi_R(v_H, p^*_H) \geq E\Pi_R(v_L, p^*_L) = \max_p E\Pi_R(v_L, p) \);

(II) Challenger \( H \) wins the primary election.

Proof. By (b) of Lemma 2, the first inequality of (I) is immediate. Thus, by the equilibrium condition (3), we obtain \( p^*_L = \hat{p}_L \), which gives us the second relationship of equality in (I).

Notice that by (a) of Lemma 2, for any \( p \in [\bar{p}, R] \), \( E\Pi_R(v_H, \hat{p}_H) > E\Pi_R(v_L, p) \). Thus, obviously \( E\Pi_R(v_H, \hat{p}_H) > E\Pi_R(v_L, p_L^*) \) and hence, \( U_H(p^*_L) \neq \emptyset \). By (a), we obtain the first inequality, \( E\Pi_R(v_H, p^*_H) \geq E\Pi_R(v_L, p^*_L) \). Combining the two relationships, we obtain (I). By a tie-breaking rule and (a) of Lemma 2, we obtain (II).

Serra (2011) defines the “primary skill bonus” by the increase of the expected campaigning skill of their nominee. As in Serra (2011), we obtain a similar result on the primary skill bonus such that challenger \( H \) always wins. Proposition 6 confirms this result.

Now, we are ready to solve for the equilibrium in the model of two stage electoral competition. Let \( \Psi(v_H, p_L) = U_H(p_L) \cap \{ d(v_H), \max \{ d(v_H), Y_H^{-1}(\bar{p}) \} \} \). Then, \( \Psi(v_H, p_L) \) is a set of policy promises for challenger \( H \) that weakly beats \( p_L \) and yields winning probability 1 in a general election.
Theorem 1. There exists a unique equilibrium \((p_L^*, p_H^*)\) such that

- \((p_L^*, p_H^*)\) is a Condorcet-winner among equilibrium policy promises;
- \(p_L^* = \hat{p}_L\);
- if \(\Psi(v_H, p_L^*) = \emptyset\), \(p_H^* = \min\{p : p \in \mathcal{U}_H(p_L^*)\}\);
- otherwise, \(p_H^* = \arg\min_{p \in \Psi(v_H, p_L^*) \cap [\bar{p}_H, R_H^*]} (R_H^* - p)\).

Proof. By Proposition 6, challenger \(H\) wins a primary election and the equilibrium condition (3) requires that challenger \(L\) maximizes median voter \(R\)'s expected payoff. By Proposition 2, this maximizer exists uniquely and thus an equilibrium policy promise \(p_L^*\) is equal to \(\hat{p}_L\) for challenger \(L\). By Lemma 2, \(\mathcal{U}_H(p_L^*) \neq \emptyset\).

If \(\Psi(v_H, p_L^*) = \emptyset\), \(\pi(v_H, p_H)\) is strictly decreasing in \(p_H \in \mathcal{U}_H(p_L^*)\). Therefore, there exists only one maximizer of \(\pi(v_H, p_H)\), which is given by \(p_H^* = \min\{p : p \in \mathcal{U}_H(p_L^*)\}\). If \(\Psi(v_H, p_L^*) \neq \emptyset\), then \(\pi(v_H, p_H) = 1\) for all \(p_H \in \Psi(v_H, p_L^*)\). By Proposition 2, median voter \(R\)'s most preferred policy is a Condorcet-winner among policy promises in \(\Psi(v_H, p_L^*)\). Then, there is no \(p_H \in \Psi(v_H, p_L^*) \cap (R_H^*, 1]\) that median voter \(R\) strictly prefers to \(R_H^*\). So, if \(R_H^* \in \mathcal{U}_H(p_L^*)\), then \(R_H^* = p_H^*\) and otherwise, \(p_H^* \in \Psi(v_H, p_L^*) \cap [\bar{p}_H, R_H^*]\) minimizes the distance to \(R_H^*\), because by Proposition 2, median voter \(R\)'s expected payoff is increasing in policy promises in \([\bar{p}_H, R_H^*]\). This completes the proof.

Theorem 1 states that challenger \(L\) would choose a maximizer of median voter \(R\)'s expected payoff in equilibrium by Lemma 2, as he cannot beat challenger \(H\) in a primary and thus he does best he can in equilibrium. On the other hand, challenger \(H\) chooses a policy promise that would be enough to beat challenger \(L\), and at the same time tries to maximize the winning probability in a general election. Thus, challenger \(H\) has to consider the position \(\bar{p}\), not just indirectly from the winning probability in maximizing the expected payoff for the median voter \(R\). In equilibrium, challenger \(H\) would directly try to maximize the winning probability itself. In the next section, we will see how this would bring the equilibrium promises away from the extreme point \(R\).

2.2 Comparative Analysis

One may wonder how the equilibrium would change without a primary election. In the literature of voting theory, Myerson (1993) compares, under different electoral systems, the incentives for candidates to create inequalities among otherwise homogeneous voters, by making promise that favor small groups, rather than appealing equally to all voters. Myerson and Weber (1993) also show the set of equilibria can vary substantially with the choice of voting
In our model, if three candidates go for an office at the same time and voters are strategic, there can be multiple equilibria and it would be difficult to compare it with ours because we would not be able to decide which equilibrium we should compare with our unique equilibrium.

Here, we will compare the policy promises that maximize the expected payoff of the median voter $R$ and the equilibrium promises. McGann (2002) studies a two-stage election where voters vote sincerely in the primary elections, namely they vote for candidates whose policies are closest to their own bliss points. Another possibility is the original maximization problem in Owen and Grofman (2006) or Roemer (1997). In their paper, they consider the problem such that given $p_{-k}, p_k$ solves}

$$
\max_p E u_R(v_k, p). \tag{16}
$$

In our model, explicitly challengers need to consider the winning probability in a general election. This is the main point that our model differs to Owen and Grofman (2006) or Roemer (1997). In this section, we consider how our model would make a difference to the solution from Problem (16). We consider how the equilibrium changes if the objective of challengers is to maximize the median voter $R$’s expected utility, rather than a winning probability in a general election. The following theorem states that relative to $R, \bar{p},$ or $\hat{p}_k$ for each $k \in \{H, L\}, p^*_H$ would become more moderate.

**Theorem 2.** The following holds:

(a) $\hat{p}_H < \hat{p}_L,$ if $\hat{p}_H \neq R$;

(b) $p^*_H < \hat{p}_H,$ if $\hat{p}_H \neq \bar{p}$;

(c) $p^*_H < \hat{p}_L.$

**Proof.** First, we show that $\hat{p}_H < \hat{p}_L.$ Notice that by Proposition (2), $\hat{p}_L \leq R$ and $\hat{p}_H \leq R.$ Then, if $\hat{p}_H \neq R,$ then the first order condition must hold. If $\hat{p}_L = R,$ the result is trivial as $\hat{p}_H < R.$ So suppose that there exists an interior solution $\hat{p}_L \in I_L.$ Then, the first order condition must hold for challenger $L,$ as well. Notice that $d(v_L) > d(v_H),$ because $g^{-1}$ is monotonically increasing. Let $p^*_H$ satisfy $F(Y_H(p^*_H)) = F(Y_L(\hat{p}_L)).$ By the definition of $Y$’s and $g$ is monotone, $p^*_H > \hat{p}_L.$ Thus, we obtain

$$
u_R(v_H, p^*_H) - u_R(v_L, \hat{p}_L) > 0. \tag{17}$$

7In a different approach, Myerson (1994) broadly argues that the foundations of any social institution may be understood in terms of more fundamental games that have multiple equilibria by using a concept of focal points proposed in Schelling (1960).
Then, by the first order condition for Challenger $H$, for $p'_H \in I_H$ and $\hat{p}_L \in I_L$, we obtain

$$\frac{d\Pi_R(v_H, p'_H)}{dp} = \pi_H(v_H, p'_H)g'(R - p'_H) - \pi'_H(v_H, p'_H)(u_R(\bar{v}, \bar{\pi}) - u_R(v_H, p'_H))$$

$$= \pi_H(v_H, p'_H)(g'(R - p'_H) + g'(R - \hat{p}_L) - g'(R - \bar{p}_L)) - \pi'_H(v_H, p'_H)\times$$

$$\times(u_R(\bar{v}, \bar{\pi}) - u_R(v_L, \hat{\pi}_L) + u_R(v_L, \hat{\pi}_L) - u_R(v_H, p'_H))$$

$$= \pi_H(v_H, p'_H)(g'(R - p'_H) - g'(R - \hat{p}_L)) + \Pi'_R(v_L, \hat{\pi}_L) + \pi'_L(v_L, \hat{\pi}_L)(u_R(v_H, p'_H) - u_R(v_L, \hat{\pi}_L))$$

$$< 0,$$

(18)

because $R - p'_H < R - \hat{p}_L$ implies $g'(R - p'_H) \leq g'(R - \hat{p}_L)$ by the convexity of $g$, and the last inequality of (18) holds by $\Pi'_R(v_L, \hat{\pi}_L) = 0$ and (17). Therefore, to reach an optimal solution for $\hat{p}_H, p'_H$ has to decrease (otherwise, by (18), $\Pi_R(v_H, p_H)$ would decrease further). So we obtain $\hat{p}_H < \hat{p}_L$ in this case.

Second, by Theorem 1, $p'_H, \hat{p}_H \in U_H(p'_L)$. Again by Theorem 1, because $p'_H$ is the minimal element of the set $U_H(p'_L)$, we obtain $\hat{p}_H \leq \hat{p}_H$. To prove the strict inequality, notice that by Lemma 2, $\Pi_R(v_H, \hat{p}_H) > \Pi_R(v_L, \hat{\pi}_L)$. Notice that $\Pi_R(v_H, \hat{p})$ is continuous in $\hat{p}$ in the interval $I_H$. If $\hat{p}_H \neq \hat{\pi}_L$, there exists a $\hat{p}_H < \hat{\pi}_H$ in the $\epsilon$-neighborhood of $\hat{p}_H$ for a sufficiently small $\epsilon$ such that

$$\Pi_R(v_H, \hat{p}_H) > \Pi_R(v_H, \hat{p}_H) > \Pi_R(v_L, \hat{\pi}_L).$$

Notice that $\hat{p}_L \in U_H(p'_L)$, and so $\hat{p}_L \geq \hat{p}_H$. Thus, we obtain (b).

To prove (c), similarly to the proof of Lemma 2, $\Pi_R(v_H, \hat{p}_L) > \Pi_R(v_L, \hat{\pi}_L)$. Therefore, we have $\hat{p}_L \in U_H(\hat{\pi}_L)$. By Theorem 1, because $p'_L$ is the minimal $\hat{\pi}_L$, and $p'_H$ is the minimal element of $U_H(p'_L)$, which indicates $p'_H \leq \hat{\pi}_L$. Then, again by $\Pi_R(v_H, \hat{\pi}_L) > \Pi_R(v_L, \hat{\pi}_L)$, we conclude that $p'_H \neq \hat{\pi}_L$. 

\[ \square \]

3 Some Examples

To quantify our result, let $g(x) = |x|$. Then, $d(v_k) = h(\bar{v}) - h(v_k) + \bar{p}$ for each $k \in \{H, L\}$.

**Proposition 7.** For each challenger $k \in \{H, L\}$ with policy promise $p_k \in [0, 1]$, the probability of winning the general elections is

$$\pi(v_k, p_k) = \begin{cases} 0 & \text{if } p_k < d(v_k) \\ 1 & \text{if } d(v_k) \leq p_k < \max \{d(v_k), 2\bar{p} - d(v_k)\} \\ 1 - F\left(\frac{d(v_k) + p_k}{2}\right) & \text{if } \max \{d(v_k), 2\bar{p} - d(v_k)\} \leq p_k < 2R - d(v_k) \\ 0 & \text{if } p_k \geq 2R - d(v_k) \end{cases}$$

where the interval $[d(v_k), \max \{d(v_k), 2\bar{p} - d(v_k)\}]$ is non-empty if and only if $h(v_k) > h(\bar{v})$.  

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Proposition 8. Suppose that \( F(x) = \frac{x - \bar{p}}{R - \bar{p}} \). Then, for each \( k \in \{H, L\} \), \( p_k^* = \hat{p}_k = R \). Moreover,

- if \( \Pi_R(v_L, R) > \Pi_R(v_H, \bar{p}_H) \), then there exists a \( p_H \in I_H \) to satisfy
  \[
  h(v_H) - R + p_H = \frac{R - d(v_L)}{2R - d(v_H) - p_H} h(v_L) - \frac{d(v_H) - d(v_L) + p_H - R}{2R - d(v_H) - p_H} (h(\bar{v}) - R + \bar{p}),
  \]

  and \( p_H^* \) is a minimal \( p_H \) to satisfy (19).

- Otherwise, \( p_H^* = \max\{\bar{p}_H, \max\{d(v_H), 2\bar{p} - d(v_H)\}\} \).

Proof. Fix \( k \in \{H, L\} \). For any \( p \in I_k \), if \( \max\{d(v_k), 2\bar{p} - d(v_k)\} \leq p < 2R - d(v_k) \),

\[
\pi'(v_k, p) = \frac{1}{2} F'(\frac{d(v_k) + p}{2}) = -\frac{1}{2} \frac{R - p}{R - \bar{p}} \pi(v_k, p) = 1 - F(\frac{d(v_k) + p}{2}) = 1 - \frac{d(v_k) - p_k - 2\bar{p}}{R - \bar{p}}.
\]

Then, we consider the first order condition:

\[
\Pi_R'(v_k, p_k) = 1 - \frac{d(v_k) + p_k - 2\bar{p}}{2(R - \bar{p})} + \frac{d(v_k) - p_k}{2(R - \bar{p})} = 1 - \frac{p_k - \bar{p}}{R - \bar{p}}.
\]

Thus the above is greater than zero if and only if \( p_k \leq R \), and equal to zero only with \( p_k = R \). By Proposition 2, because \( R \)'s expected payoff is single-peaked, we obtain \( \hat{p}_k = R \).

Next, observe that

\[
\Pi_R(v_L, R) = F(\frac{d(v_L) + R}{2}) (h(\bar{v}) - R + \bar{p}) + (1 - F(\frac{d(v_L) + R}{2})) h(v_L)
\]

\[
\Pi_R(v_H, p_H) = F(\frac{d(v_H) + p_H}{2}) (h(\bar{v}) - R + \bar{p}) + (1 - F(\frac{d(v_H) + p_H}{2})) (h(v_H) - R + p_H).
\]

Because \( \Pi_R(v_H, p_H) \) is continuous in \( I_H \), by the intermediate value theorem, if

\[
F(\frac{d(v_L) + R}{2}) (h(\bar{v}) - R + \bar{p}) + (1 - F(\frac{d(v_L) + R}{2})) h(v_L) > \Pi_R(v_H, \bar{p}_H),
\]

By substituting the above into Proposition 1, we obtain the desired result.
together with Lemma 2, there exists a $p_H^*$ that satisfies
\[
\text{EI}_R(v_H, p_H^*) = \frac{d(v_L) + R - 2\bar{p}}{2(R - \bar{p})} (h(\bar{v}) - R + \bar{p}) + \frac{R - d(v_L)}{2(R - \bar{p})} h(v_L). \tag{20}
\]

Now, (20) can be rewritten as the RHS of (19). Otherwise, $\text{EI}_R(v_H, p_H) > \text{EI}_R(v_L, R)$. Then, if \( \max \{d(v_H), 2\bar{p} - d(v_H)\} \) \( \in I_H \), namely \( \max \{d(v_H), 2\bar{p} - d(v_H)\} > \bar{p}_H \), then $p_H^* = \max \{d(v_H), 2\bar{p} - d(v_H)\}$ so that $\pi(v_H, p_H^*) = 1$ and $p_H^*$ is the Condorcet winner among the policy promises that yield the winning probability 1 for challenger $H$. Otherwise, $p_H^* = \bar{p}_H$. Thus, we obtain $p_H^* = \max \{\bar{p}_H, \max \{d(v_H), 2\bar{p} - d(v_H)\}\}$.

**Corollary 1.** Suppose that $F(x) = \frac{x - \bar{p}}{R - \bar{p}}$. If $p_H^*$ satisfies (19) with equality, then $p_H^*$ converges from below to $R$, as $h(v)$ converges to 0 for any $v$.

**Proof.** Suppose that $F(x) = \frac{x - \bar{p}}{R - \bar{p}}$. If $p_H^*$ satisfies (19) with equality, it is easy to check that $p_h^*$ converges to $R$. Then,
\[
F\left(\frac{h(\bar{v}) - h(v_H)}{2(R - \bar{p})}\right) = \frac{1}{2} + \frac{h(\bar{v}) - h(v_H)}{2(R - \bar{p})}. \tag{21}
\]

Let $p_H = R$. Then, the LHS of (19) is $h(v_H)$. The RHS is
\[
\frac{R - d(v_L)}{R - d(v_H)} h(v_L) + \frac{h(v_H) - h(v_L)}{R - d(v_H)} \cdot (h(\bar{v}) - R + \bar{p}).
\]

Note that $R - d(v_L) + h(v_H) - h(v_L) = R - h(\bar{v}) + h(v_H) - \bar{p} = R - d(v_H)$. Thus by Assumption (4), the RHS is smaller than $h(v_L)$. Thus, we conclude that the LHS is strictly greater than the RHS, which implies that $p_h^*$ converges to $R$ from below.

It is interesting to see $p_h^*$ is not equal to $R$ even when utility from valence is extremely small. In a sense, a primary causes a more moderate policy to be selected and increases the probability that a candidate of higher valence wins.

As mentioned before, our method still works when there is a point that is not twice-differentiable. In the next example, we show this and also show that when the likelihood of the median voter’s position increases up to some point and then decreases, challenger $L$ needs to take into account of $d(v_L)$ to his position in equilibrium.

**Example 1.** Let $\bar{p}_L = 0.5$, and $\bar{p}_H = 0.4$. Suppose that $\bar{p} = 0.2$, $R = 0.8$, and $h(v_L) = 0.2$, $h(\bar{v}) = 0.5$, $h(v_H) = 0.8$. Suppose that
\[
F(x) = \begin{cases} 
2 \times \frac{(x - \bar{p})^2}{(R - \bar{p})^2} & \text{for } x \in (\bar{p}, \frac{\bar{p} + R}{2}); \\
1 - 2 \times \frac{(R - x)^2}{(R - \bar{p})^2} & \text{for } x \in (\frac{\bar{p} + R}{2}, R].
\end{cases}
\]

Then, $\bar{p}_L = p_h^* = 0.5$, $\bar{p}_H = 0.8$, and $p_h^* = 0.5$ with $\pi(v_H, p_H) = 1$ for all $p_H \in [\bar{p}_H, 0.5]$. 

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Proof. We obtain the first derivative $F'$ as follows:

$$F'(x) = \begin{cases} 
4 \times \frac{(x-\bar{p})}{(R-p)^2} & \text{for } x \in \left(\bar{p}, \frac{\bar{p}+R}{2}\right]; \\
4 \times \frac{(R-x)}{(R-p)^2} & \text{for } x \in \left(\frac{\bar{p}+R}{2}, R\right].
\end{cases}$$

By calculation, we can show that $\frac{F'(x)}{1-F(x)}$ is increasing for $x \in [\bar{p}, R]$. Without loss of generality, fix $k \in \{H, L\}$. For some $p \in I_k$ and with $\max \{d(v_k), 2\bar{p} - d(v_k)\} \leq p < 2R - d(v_k)$, if $\frac{d(v_k)+p}{2} \leq \frac{\bar{p}+R}{2}$,

$$\begin{align*}
\pi'(v_k, p) &= -\frac{1}{2} F'(\frac{d(v_k)+p}{2}) = -2 \times \frac{d(v_k)+p}{2(R-p)^2}, \\
\pi(v_k, p) &= 1 - F(\frac{d(v_k)+p}{2}) = 1 - 2 \times \frac{d(v_k)+p}{2(R-p)^2},
\end{align*}$$

and if $\frac{d(v_k)+p}{2} > \frac{\bar{p}+R}{2}$,

$$\begin{align*}
\pi'(v_k, p) &= -\frac{1}{2} F'(\frac{d(v_k)+p}{2}) = -2 \times \frac{R-d(v_k)+p}{2(R-p)^2}, \\
\pi(v_k, p) &= 1 - F(\frac{d(v_k)+p}{2}) = 2 \times \frac{R-d(v_k)+p}{2(R-p)^2}.
\end{align*}$$

First, suppose that $\frac{d(v_k)+p_k}{2} \leq \frac{\bar{p}+R}{2}$. Note that

$$\begin{align*}
\text{EII}'_R(v_k, p_k) &= \left(1 - 2 \times \frac{\frac{d(v_k)+p_k}{2} - \bar{p}}{(R-p)^2}\right) - 2 \times \frac{\frac{d(v_k)+p_k}{2} - \bar{p}}{(R-p)^2} \cdot (p_k - d(v_k)) \\
&= 1 - 2 \times \frac{\frac{d(v_k)+p_k}{2} - \bar{p}}{(R-p)^2} \left(\frac{d(v_k)+p_k}{2} - \bar{p} + p_k - d(v_k)\right) \\
&= 1 - \frac{d(v_k)+p_k-\bar{p}}{(R-p)^2} \left(3p_k-d(v_k)-2\bar{p}\right). \tag{22}
\end{align*}$$

Thus, $\text{EII}'_R(v_k, p_k) > 0$. Then, $\text{EII}_R(v_k, p_k)$ takes on a maximand when $p_k = \min\{\bar{p} + R - d(v_k), R\}$ for each $k \in \{H, L\}$. Now, by substituting the values for each parameter, $d(v_H) = -0.1$ and $d(v_L) = 0.5$. So, in the first case, we obtain our candidate solution $\hat{p}_H = 0.8$ and $\hat{p}_L = 0.5$ by Proposition 2.

Second, suppose that $\frac{d(v_k)+p_k}{2} > \frac{\bar{p}+R}{2}$. Then, the first order condition yields

$$\begin{align*}
\text{EII}'_R(v_k, p_k) &= 2 \times \frac{(R-d(v_k)+p_k)^2}{(R-p)^2} + 2 \times \frac{R-d(v_k)+p_k}{(R-p)^2} \times (d(v_k) - p_k) \\
&= 2 \times \frac{R-d(v_k)+p_k}{(R-p)^2} \times \left(R - \frac{d(v_k)+p_k}{2} + d(v_k) - p_k\right).
\end{align*}$$

When $\text{EII}'_R(v_k, p_k) = 0$, since $(R - \frac{d(v_k)+p_k}{2}) > 0$,

$$2R - d(v_k) - p_k + 2d(v_k) - 2p_k = 0.$$

Then, we obtain $\hat{p}_k = \frac{2R+d(v_k)}{5}$, which implies $\hat{p}_H = 0.5$ and $\hat{p}_L = 0.7$. However, $\hat{p}_H = 0.5$ is impossible because $\frac{\hat{p}_H + d(v_H)}{2} < \frac{\bar{p}+R}{2}$. Thus, we conclude $\hat{p}_H = 0.8$. By Proposition 2 and by the first case, we conclude $\hat{p}_L = 0.7$. 

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Finally, we are interested in $p_H \in I_H$ that maximizes $\pi(v_H, p_H)$ with satisfying

$$\Pi_R(v_H, p_H) \geq \Pi_R(v_L, 0.7) = 0.1 \times \frac{5}{9}.$$ 

Now, $\max\{d(v_H), Y_H^{-1}(0.2)\} = \max\{-0.1, 0.5\} = 0.5$. As $F(\frac{-0.1+0.5}{2}) = F(0.2) = 0$, $\pi(v_H, 0.5) = 1$. Thus, as $p_H = 0.4$, $p_H = 0.5$. \hfill \square

**Example 2.** Suppose that $F$ and $\bar{p}_H, \bar{p}_L$ are defined as in Example 1. Suppose that $\bar{p} = 0.2, R = 0.8$, and $h(v_L) = 0.2, h(v_H) = 0.5, h(\bar{v}) = 0.8$. Then, $\hat{p}_H = 0.7, \hat{p}_L = 0.8$ and $p_H^p = 0.5$ with $\pi(v_H, 0.5) = 0.5$.

**Proof.** Then, $d(v_L) = 1.0 - 0.2 = 0.8$ and $d(v_H) = 1.0 - 0.5 = 0.5$. By Example 2, $\Pi_R(v_k, p_k)$ takes on a maximand when $p_k = \min\{\bar{p} + R - d(v_k), R\}$ for each $k \in \{H, L\}$. Now, by substituting the values for each parameter, $d(v_H) = -0.1$ and $d(v_L) = 0.5$. So, in the first case, we obtain our candidate solution $\hat{p}_H = 0.8$ and $\hat{p}_L = 0.5$.

In the second case,

\[
\hat{p}_H = \frac{2R + d(v_H)}{3} = \frac{2 \times 0.8 + 0.5}{3} = 0.7
\]

\[
\hat{p}_L = \frac{2R + d(v_L)}{3} = \frac{2 \times 0.8 + 0.8}{3} = 0.8.
\]

In the second case, it must be $\frac{d(v_k) + p_k}{2} > \frac{\bar{p} + R}{2}$. As $\frac{0.8 + 0.8}{2} > 0.5$ and $\frac{0.5 + 0.7}{2} > 0.5$, $\hat{p}_H = 0.7$ and $\hat{p}_L = 0.8$ satisfy the constraints. By Proposition 2, we obtain $\hat{p}_H = 0.7$ and $\hat{p}_L = 0.5$.

Then, notice that as $\hat{p}_L = d(v_L), \pi(v_L, 0.5) = 1$. We are interested in $p_H$ to satisfy $\Pi_R(v_H, p_H) \geq \Pi_R(v_L, 0.5) = 0.2$. By Proposition 2, $\Pi_R(v_H, p) < \Pi_R(v_H, 0.5) = 0.2$ holds for $p \in [\hat{p}_H, 0.5)$. Thus, we obtain $p_H = 0.5$ and $\pi(v_H, 0.5) = 0.5$. \hfill \square

### 3.1 Discussion and the Empirical Literature

First, we consider how primary can increase the winning chance for the challenging party. As a corollary to Theorem 2, we obtain the following result.

**Corollary 2.** It is held that $\pi(v_H, p_H^* ) \geq \pi(v_H, \hat{p}_H)$ with the equality only if $\pi(v_H, p_H^* ) = \pi(v_H, \hat{p}_H) = 1$.

An implication of this corollary is that unless the challenging party has a very strong candidate, for which $\pi(v_H, p_H^* ) = \pi(v_H, \hat{p}_H) = 1$ holds, the challenging party obtains a higher chance of winning in a general election by using a primary election. This result is consistent with the empirical finding in Carey and Polga-Hecimovich (2006) that primary elections can select candidates who are stronger than those selected by other procedures. Moreover, the condition of $\pi(v_H, p_H^* ) = \pi(v_H, \hat{p}_H) = 1$ indicates that primaries tend to benefit weaker parties, which is also numerically shown by Adams and Merrill (2008). They label this effect
the weaker party’s competitive primary advantage. Corollary 2 also implies that through primaries, weaker parties’ candidates can select policies that is more competitive to the incumbent candidate in the general election.

Second, we study the relationship between the primary advantage and the salience of valence compared to policy issues to voters. On the one hand, Adams and Merrill (2008) find that decreasing the salience of valence compared to policy should make primaries less likely to be adopted. On the other hand, Serra (2011) shows that as valence has less weight in voters; preferences compared to policy, primaries become certain, which is opposite to the finding by Adams and Merrill (2008). In our model, primaries are assumed to happen, although we can interpret the advantage of having a primary in our model is positively correlated to the likelihood of primaries being held, because as this advantage is larger, party’s leaders would benefit more from primaries.

To measure a magnitude of this advantage explicitly, we define a primary advantage by a function $S : \mathbb{R} \times [0, 1] \times [0, 1] \to \mathbb{R}$ such that

$$S(v_H, p_H^*, \hat{p}_H) = \pi(v_H, p_H^*) - \pi(v_H, \hat{p}_H).$$

Then, $S(v_H, p_H^*, \hat{p}_H)$ measures the benefit that party leaders’ obtain via the first stage primary of electoral competition. In other words, the party’s winning chance increases by $S(v_H, p_H^*, \hat{p}_H)$.

In the literature, Serra (2011) finds an opposite result to Adams and Merrill (2008) on the likelihood of holding primaries. The following analysis provides both of the results in the literature, which looks contradictory to each other. To convey a simple intuition, we take the assumptions of uniform distribution $F$ under Proposition 8 and $g(x) = x$ so that we can focus on the effects of valences on winning chances. Moreover, suppose that challenger $H$ has a reasonably high valence so that $\pi(v_H, p_H^*) = 1$ holds. To focus on the simplest case, we make this assumption. Without this assumption, (20) holds and then $\pi(v_H, p_H^*) = 1 - F(\frac{d(v_H) + p_H^*}{2})$ where $p_H^*$ satisfies (19). As we see from (19), then the prediction on the likelihood of holding primaries would be more complicated.

In our model, the benefit from primaries rests on the shape of $h$ as well as the distribution of the overall median voter’s bliss points $F$. Then, by substituting $F$ from (21), the primary advantage is given by

$$S(v_H, p_H^*, \hat{p}_H) = \pi(v_H, p_H^*) - \pi(v_H, R) = \frac{1}{2} - \frac{h(\bar{v}) - h(v_H)}{2(R - \bar{p})}. \quad (23)$$

Let $h_1, h_2 : \mathbb{R} \to [0, 1]$ denote a utility from valence in voters’ utility function, $S_1, S_2 : \mathbb{R} \times [0, 1] \times [0, 1] \to \mathbb{R}$ denote primary advantages corresponding to $h_1, h_2$, respectively. Moreover, let $p_H^*, \tilde{p}_H$ denote $p_H^*, \hat{p}_H$ corresponding to $h_l$ for each $l \in \{1, 2\}$.

Now, we calculate the difference of the two primary bonuses. This difference shows the difference of likelihood for holding primaries, because one at $h_2$ is bigger than the other at $h_1$, primaries are more likely to be held at $h_2$, which gives larger advantage than the one at $h_1$. 20
Lemma 3. Suppose that \( g(x) = x, \) \( F(x) = \frac{x^p}{R^p} \) and that \( v_H \) is reasonably high. Then, the difference of the two primary advantages, \( S_2(v_H, p_2^*, \hat{p}_2) - S(v_H, p_1^*, \hat{p}_1) \) is given by \( \frac{h_1(\hat{v}) - h_2(\hat{v})}{2(R-p)} \).

Proof. By (23), the difference of the two primary advantages is
\[
S_2(v_H, p_2^*, \hat{p}_2) - S(v_H, p_1^*, \hat{p}_1) = \frac{h_1(\hat{v}) - h_1(\bar{v})}{2(R-p)} - \frac{h_2(\hat{v}) - h_2(\bar{v})}{2(R-p)}.
\]

The following result compares the difference between the two primary advantages.

Proposition 9. Suppose that \( h_2(v) > h_1(v) \) for all \( v \in \mathbb{R} \). Suppose that \( g(x) = x, \) \( F(x) = \frac{x^p}{R^p} \) and that \( v_H \) is reasonably high. Then,

1. when \( h_2(\hat{v}) - h_1(\hat{v}) < h_2(\bar{v}) - h_1(\bar{v}) \), the primary advantage at \( h_2 \) is higher than the one at \( h_1 \);

2. when \( h_2(\hat{v}) - h_1(\hat{v}) > h_2(\bar{v}) - h_1(\bar{v}) \), the primary advantage at \( h_1 \) is higher than the one at \( h_2 \).

The case such that \( h_1(\hat{v}) = \lambda_1 v \) and \( h_2(\hat{v}) = \lambda_2 v \) for some positive constants \( \lambda_1 < \lambda_2 \) is a special case for (1). The extreme case of this situation is the case that corresponds to Serra (2011). As voters care about valence more and more as valence increases and a primary selects challenger \( H \), a primary is more likely to happen. However, in case 2, at \( h_2 \) voters care more about valence being reasonable, and after this reasonable point, a marginal utility that voter can obtain from marginal increase of valence decreases. Thus, a primary becomes less likely, because increasing a chance for the Challenger \( H \) does not buy much for voters’ utility. A similar result is presented in Adams and Merrill (2008). Corollary 9 comprehends both of the results in the literature, which looks contradictory to each other.

Finally, we investigate the relationship between valences, policy promises and winning chances. Unlike in the previous literature, our model is two-stage election. So, our analysis takes into account of the three valences of the incumbent, and the two candidates in the challenging party. To do so, let \( P : \mathbb{R} \to (\bar{p}, R) \) to satisfy
\[
\left( \frac{h(\hat{v}) - h(\bar{v})}{2(R-p)} + \frac{1}{2} \right) (h(\hat{v}) - R + \bar{p}) + \left( \frac{1}{2} - \frac{h(\hat{v}) - h(\bar{v})}{2(R-p)} \right) h(\bar{v}) \]
\[
= \left( \frac{h(\hat{v}) - h(\bar{v})}{2(R-p)} + \frac{P(v_H) - \bar{p}}{2(R-p)} \right) (h(\hat{v}) - R + \bar{p})
\]
\[
+ \left( \frac{1}{2} - \frac{P(v_H) - \bar{p}}{2(R-p)} - \frac{h(\hat{v}) - h(\bar{v})}{2(R-p)} \right) (h(v_H) - R + P(v_H)).
\]

Then, there exists a \( p_H^* = P(v_H) \) to satisfy (20), and by implicit function theorem, \( P \) is continuously differentiable. We characterize how a policy bias \( R - P(v_H) \) or a winning probability \( \pi(v_H, P(v_H)) \) changes when \( v_H \) changes in the next proposition.
Proposition 10. Suppose that $g(x) = x$ and $F(x) = \frac{x - \bar{p}}{R - \bar{p}}$. Take $\bar{v}_H$ and suppose that (20) holds in the $\epsilon$-neighborhood of $\bar{v}_H$ for sufficiently small $\epsilon > 0$. Then,

(A) $\frac{\text{dln}(R - P(v_H))}{\text{d}v_H} = h'(v_H)(h(v_H) - h(\bar{v}) + R - \bar{p})$;

(B) $\frac{\text{d}x(v_H, P(v_H))}{\text{d}v_H} = \frac{h'(v_H)}{2(R - \bar{p})} \cdot (1 - (R - P(v_H))(h(v_H) - h(\bar{v}) + R - \bar{p}))$.

Proof. Take the first derivative of (25) with respect to $v_H$, we obtain

$$0 = \left(-\frac{h'(v_H)}{2(R - \bar{p})} + \frac{\text{d}P(v_H)}{\text{d}v_H} \right)(h(\bar{v}) - R + \bar{p}) - \frac{(h'(v_H)(h(v_H) - R + P(v_H)))}{2(R - \bar{p})}$$

$$+ \left(1 - \frac{P(v_H) - \bar{p}}{2(R - \bar{p})} - \frac{h'(v_H)}{2(R - \bar{p})}\right) \left(h'(v_H) + \frac{\text{d}P(v_H)}{\text{d}v_H}\right).$$

Arranging orders in (26), we obtain

$$\frac{\text{d}P(v_H)}{\text{d}v_H} \cdot \left(P(v_H) - \bar{p} - 1\right) = h'(v_H)\left(\frac{h(v_H) - h(\bar{v})}{R - \bar{p}} + 1\right).$$

Organizing terms in (27), we obtain (A). Second, notice that

$$\frac{\text{d}x(v_H, P(v_H))}{\text{d}v_H} = -\frac{1}{2} F'' \left(\frac{\text{d}P(v_H)}{\text{d}v_H}\right)$$

$$= -\frac{1}{2} \frac{h'(v_H)}{2(R - \bar{p})} \cdot \left(-h'(v_H) + \frac{\text{d}P(v_H)}{\text{d}v_H}\right)$$

$$= \frac{h'(v_H)}{2(R - \bar{p})} \cdot \left(1 - (R - P(v_H))\right) \left(h'(v_H) - h(\bar{v}) + R - \bar{p}\right).$$

Proposition 10 indicates that in our model, even in the presence of primary elections, a similar result to the empirical findings in Green and Hobolt (2008) or Clark and Leiter (2013) holds. The interpretation of this result is that as both parties’ median voters’ distance decreases, then the effect of valence on policy decreases and then the effect of valence on winning chances increases. Moreover, Proposition 10 also has an implication that is consistent with the finding in Buttice and Stone (2012). Although our settings are different in that we consider a presidential election and Buttice and Stone (2012) studies US House elections, we can interpret Proposition 10 as saying that as $R - \bar{p}$ increases, the effect of valence on winning chances decreases. Proposition 10 provides another conjecture that as the ideological difference decreases, valence becomes more important on electoral outcome, even in the presence of primary elections.

4 Concluding Remarks

In this paper, we have proposed a simple model of two-stage election with a primary and a general. By allowing voters to care both about policy promises and valence, we showed that
the existence of primary elections brings out a more moderate policy promise and at the same
time, increases the probability of a candidate with valence to win. Our model is simple and
tractable. At the same time, it is stable enough to lend itself to various extensions.

There are a number of directions in which our model can be brought. One direction is
spoiler voters. Chen and Yang (2002) studies an open primary election with two competing
candidates and conclude that the spoiler voter effect on the primary outcome depends on the
party sizes and turn out rates. Studying the effect of spoiler voters on the equilibrium outcome
within our framework might be interesting.

Another direction is that introducing a certain type of errors to the median voter $R$’s po-
sition. In this paper, we have assumed that it is publicly known and exogenously given. This
may be a reasonable assumption in a closed primary but as party size increases and more
people join the primary, there may be some uncertainty about it. It would require a careful
consideration on modelling the interaction of two distributions for overall median voter and
this error. Although technically it might be challenging, it may bring realistic predictions on
the equilibrium outcome in open primaries.

Moreover, in this paper, we have assumed that valence and a policy promise are indepen-
dent and voters’ utility is a function of the two variables. In the original model of Roemer
(1997), it is assumed that a policy promise is a tax rate to fund public goods and another vari-
able is an amount of funded public goods. Although our settings are quite different, Anh and
Oliveros (2012) studies elections that simultaneously decide multiple issues, where voters have
independent private values over bundles of issues, while Lizzeri and Persico (2001) studies the
public goods provision under various electoral incentives. It is certainly interesting to see what
would be an equilibrium outcome and who would benefit from the existence of primaries by
bringing our model to the original setting in Roemer (1997).

Furthermore, Adams and Merrill (2008) have developed a model of policy-seeking parties
in a parliamentary democracy competing in a PR electoral system, in which party elites are un-
certain about voters evaluations of the parties valence attributes such as competence, integrity
and charisma. Adams et al. (2013) extends that model to situations where voters hold coaliti-
tions of parties collectively responsible for their valence-related performances. In our paper, it
is assumed that the valence values are exogenously given. It is interesting to add uncertainty
about valences and consider the aspect of collective responsibility for it as in Adams et al.
(2013).
Appendix: Lemmas and Proofs

Lemma 4. If $M \geq p_k \geq d(v_k)$ for some $k \in \{H, L\}$, then
\[
h(\bar{v}) - h(v_k) \leq g(M - \bar{p}) - g(M - p_k).
\]

Proof of Lemma 4. As $M \geq p_k \geq d(v_k)$ and $g$ is increasing,
\[
g(M - \bar{p}) - g(M - p_k) \geq g(M - \bar{p}) - g(M - p_k - g^{-1}(h(\bar{v}) - h(v_k))). \quad (29)
\]
Then, as $g$ is convex, and $M - \bar{p} \geq g^{-1}(h(\bar{v}) - h(v_k))$,
\[
\frac{g(M - \bar{p}) - g(M - \bar{p} - g^{-1}(h(\bar{v}) - h(v_k)))}{g^{-1}(h(\bar{v}) - h(v_k))} \geq \frac{h(\bar{v}) - h(v_k) - g(0)}{g^{-1}(h(\bar{v}) - h(v_k))}. \quad (30)
\]
By combining (29) and (30), we obtain the desired result. \hfill \square

Lemma 5. If $h(\bar{v}) - h(v_k) > -g(p_k - \bar{p})$ and $h(\bar{v}) - h(v_k) < g(p_k - \bar{p})$ for some $k \in \{H, L\}$ and some $p_k > \bar{p}$, then $Y_k$ is continuously differentiable, $\bar{p} \leq Y_k(p_k) \leq p_k$, and $0 < Y'_k(p_k) < 1$.

Proof of Lemma 5. Notice that $g(x - \bar{p}) - g(p_k - x)$ is continuous and monotonically increasing in $x \in [\bar{p}, p_k]$. Thus, by intermediate value theorem, when the two conditions hold, there exists a desired $Y_k(p_k)$ for each $p_k \in [\bar{p}, R]$. By implicit function theorem, because $g$ is continuously differentiable, $Y_k$ is also continuously differentiable. By taking the first derivative with respect to $p_k$, we obtain
\[
0 = g'(Y_k(p_k) - \bar{p})Y'_k(p_k) - g'(p_k - Y_k(p_k))(1 - Y'_k(p_k)).
\]
As $g$ is strictly increasing, $Y'_k(p_k) \neq 0, 1$. Then,
\[
g'(Y_k(p_k)(1 - Y'_k(p_k))) = g'(Y_k(p_k) - \bar{p})Y'_k(p_k).
\]
By taking the second derivative, the LHS is
\[
g''(p_k - Y_k(p_k))(1 - Y'_k(p_k))^2 - g'(p_k - Y_k(p_k))Y''_k(p_k)
\]
and the RHS is
\[
g''(Y_k(p_k) - \bar{p})(Y'_k(p_k))^2 + g'(Y_k(p_k) - \bar{p})Y''_k(p_k).
\]
Thus,
\[
g''(p_k - Y_k(p_k))(1 - Y'_k(p_k))^2 - g''(Y_k(p_k) - \bar{p})(Y'_k(p_k))^2
\]
\[
= (g'(Y_k(p_k) - \bar{p}) + g'(p_k - Y_k(p_k))) Y''_k(p_k).
\]
\hfill \square
Lemma 6. Challenger $k$ wins in a general election if (1) $d(v_k) < p_k \leq M$, or (2) $Y_k(p_k) \leq M \leq p_k$.

Proof of Lemma 6. Fix $k \in \{H, L\}$ and let $M$ be a realized overall median voter’s position. Case 1: $M \geq p_k$. The result is direct from Lemma 4.

Case 2: $M < p_k$. Since $g$ is monotonically increasing, if $M \geq Y_k(p_k)$, challenger $k$ wins.  

Lemma 7. Suppose that $h(\bar{v}) - h(v_k) = -g(p_k - \bar{p})$ and $h(\bar{v}) - h(v_k) < g(p_k - \bar{p})$ for some $k \in \{H, L\}$ and some $p_k > \bar{p}$. If $p_k < d(v_k)$, then there is no $Y_k(p_k) \in [\bar{p}, p_k]$ to satisfy

$$h(\bar{v}) - h(v_k) = g(Y_k(p_k) - \bar{p}) - g(p_k - Y_k(p_k)).$$

Proof of Lemma 7. Notice that

$$g(d(v_k) - \bar{p}) - g(d(v_k) - d(v_k)) = h(\bar{v}) - h(v_k).\quad (31)$$

Take $p_k < d(v_k)$. Then, suppose that there is $Y_k(p_k) \in [\bar{p}, p_k]$ to satisfy

$$h(\bar{v}) - h(v_k) = g(Y_k(p_k) - \bar{p}) - g(p_k - Y_k(p_k)).\quad (32)$$

Then, by combining (31) and (32),

$$g(d(v_k) - \bar{p}) - g(d(v_k) - d(v_k)) = g(Y_k(p_k) - \bar{p}) - g(p_k - Y_k(p_k)).\quad (33)$$

As $d(v_k) - \bar{p} > Y_k(p_k) - \bar{p}$ and $g(p_k - Y_k(p_k)) > 0$, (33) is impossible.  

Proof of Proposition 1. Fix $k \in \{H, L\}$. By Lemma 6,

$$\pi(v_k, p_k) = \Pr(M \geq Y_k(p_k) \mid M \leq p_k) \cdot \Pr(M \leq p_k) + \Pr(d(v_k) \leq p_k \mid M > p_k) \cdot \Pr(M > p_k).$$

By the Bayes’ rule,

$$\Pr(M \geq Y_k(p_k) \mid M \leq p_k) = \frac{\Pr(Y_k(p_k) \leq M \leq p_k)}{\Pr(M \leq p_k)}.$$

(34)

Define the indicator function

$$1(p_k \geq d(v_k)) := \Pr(p_k \geq d(v_k) \mid M > p_k) = \begin{cases} 1 & \text{if } p_k \geq d(v_k) \\ 0 & \text{otherwise.} \end{cases}$$

We can therefore restate $\pi(v_k, p_k)$ as

$$\pi(v_k, p_k) = \Pr(Y_k(p_k) \leq M \leq p_k) + 1(p_k \geq d(v_k)) \cdot \Pr(M > p_k).$$

Then, $\pi(v_k, p_k) = 0$ if $p_k < d(v_k)$, because Lemma 6 and Lemma 7 indicate that the set \{M \in [0, 1] : Y_k(p_k) \leq M \leq p_k\} is empty.
Then,
\[
\pi(v_k, p_k) = \begin{cases} 
1 - F(Y_k(p_k)) & \text{if } p_k \geq d(v_k) \\
0 & \text{otherwise}
\end{cases}.
\]

The rest of the cases are easily derived because
\[
F(m) = \begin{cases} 
0 & \text{for all } m \leq \bar{p} \\
1 & \text{for all } m \geq R,
\end{cases}
\]
such that for all \( p_k \geq d(v_k) \):
\[
F(Y_k(p_k)) = \begin{cases} 
0 & \text{for all } Y_k(p_k) \leq \bar{p} \\
1 & \text{for all } Y_k(p_k) \geq R.
\end{cases}
\]

Finally, as \( Y_k \) is continuously differentiable and monotone, \( Y_k^{-1} \) is well-defined in the interval \([\bar{p}, p_k]\) for each \( p_k \in [\bar{p}, R] \). By (31), we know that \( Y_k(d(v_k)) = d(v_k) \). So, if \( Y^{-1}(d(v_k)) < Y^{-1}(\bar{p}) \), then the interval \([d(v_k), \max\{d(v_k), Y_k^{-1}(\bar{p})\}]\) is not empty. Thus, it is not empty if and only if \( \bar{p} \geq d(v_k) \). This completes the proof. \( \square \)
References


